Introduction to the p-adic numbers
Exercise Sheet 4

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This exercise sheet is split into sections:

A extremely concrete computations to help unpack definitions;
B theoretical questions which use only major results/definitions in the course;
C theoretical results requiring some thought.

The recommended approach is to focus primarily on sections B and C once you are comfortable. You should only answer questions in section A where you're not confident with the definitions of the objects involved.

Section A

1. Compute the Mahler expansion for each of the following continuous functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ (for all p):
   (a) $f(X) = X^7 + 1$
   (b) $f(X) = X^5 + X + 3$
   (c) $f(X) = X^4 + X^3 + X^2 + X + 1$
2. Recall the continuous function $\mathbf{a_0} : \mathbb{Z}_p \to \mathbb{Q}_p$, defined on p-adic expansions by $\mathbf{a_0}\left(\sum_{k=0}^{\infty} a_k p^k\right) = a_0$. Compute the first 3 terms of the Mahler expansion of $\mathbf{a_0}$ for:
   (a) $p = 2$
   (b) $p = 3$
   (c) $p = 5$
3. Draw (an approximation of)
   (a) $\mathbb{Z}_5$;
   (b) $5\mathbb{Z}_5 = \{x \in \mathbb{Z}_5 \ : \ x \equiv 0 \mod 5\}$;
   (c) $1 + 5\mathbb{Z}_5 = \{x \in \mathbb{Z}_5 \ : \ x \equiv 1 \mod 5\}$.

Section B

4. Show that for every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ we have
$$(\Delta^n f)(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k).$$
5. Let $f(x) = \frac{1}{(x^2+1)}$.
   (a) Show that $f(x)$ is defined for all $x \in \mathbb{Z}_3$.
   (b) Show that $f(x)$ is continuous as a function $\mathbb{Z}_3 \to \mathbb{Q}_3$.
   (c) Determine the first 4 coefficients in the Mahler expansion of $f(x)$.

6. (Lemma 5.18) Show that for every $n \geq 0$ and every $\alpha \in \mathbb{Z}_p$ we have

$$\binom{\alpha}{n} \in \mathbb{Z}_p.$$

[*Hint: look back at how you did Q6 on sheet 1!*]

7. Show that if $f(x)$ is a polynomial of degree d with Mahler expansion $f(x) = \sum_{n=0}^{\infty} \alpha_n \binom{x}{n}$ then for all $n \geq d+1$ we have $\alpha_n = 0$.

8. (Lemma 5.10) Prove that if $X \subseteq \mathbb{Q}_p$ and $f, g : X \to \mathbb{Q}_p$ are continuous functions then both the sum $f+g$ and product fg are continous.

Section C

9. Let $f(x) = \frac{1}{(x^4+1)}$, and let p be a prime number such that $p \not\equiv 1 \mod 8$.
   (a) Show that $f(x)$ defined for all $x \in \mathbb{Z}_p$.
   (b) Show that $f(x)$ is continuous as a function $\mathbb{Z}_p \to \mathbb{Q}_p$.
   (c) Determine the first 4 coefficients of the Mahler expansion of $f(x)$.

10. Let $\alpha \in \mathbb{Z}_p$, then define a continuous function of $x \in \mathbb{Z}_p$ which is equal to $(1+p\alpha)^x$ when $x \in \mathbb{Z}$.

11. We say that a metric space X is *totally disconnected* if every pair of elements can be separated by open sets. That is, for every pair of elements $x \neq y \in X$ there exist open sets $U_x, U_y \subseteq X$ such that the following all hold:

    - $U_x \cap U_y = \emptyset$;
    - $x \in U_x$ and $y \in U_y$;
    - $U_x \cup U_y = X$.

    Show that $\mathbb{Q}_p$ is totally disconnected.

12. We say that a metric space X is *compact* if every open cover of X has a finite subcover. That is, for every collection of open subsets $(U_i)_{i \in I}$ satisfying $\cup_{i \in I} U_i = X$, there is a finite subset $J \subseteq I$ such that $X = \cup_{i \in J} U_i$.
    (a) Prove that $\mathbb{Z}_p$ is compact.
    (b) Show that $\mathbb{Q}_p$ is not compact.