Introduction to the p-adic numbers
Exercise Sheet 2

Ross Paterson

This exercise sheet is split into three sections:

A extremely concrete computations to help unpack definitions;
B theoretical questions which use only major results/definitions in the course;
C theoretical results requiring some thought.

The recommended approach is to focus primarily on sections B and C once you are comfortable. You should only answer questions in section A where you're not confident with the definitions of the objects involved.

Section A

1. Find the first 3 non-zero terms in the 5-adic expansion of $x + y$ for each of the following
   (a) $x = -3$ and $y = \frac{32}{25}$.
   (b) $x = \frac{1}{3}$ and $y = 156$.
2. Find the first 3 non-zero terms in the 5-adic expansion of xy for each of the following
   (a) $x = -3$ and $y = \frac{32}{25}$.
   (b) $x = \frac{1}{3}$ and $y = 156$.

Section B

3. Which of the following sequences converge in $\mathbb{Q}_p$?
   (a) $\left(\frac{1}{n}\right)_n$
   (b) $(n!)_n$
   (c) $(np^n + 2p)_n$
4. Which of the following series are convergent in $\mathbb{Q}_p$?
   (a) $\sum_{n=1}^{\infty} n!$.
   (b) $\sum_{n=1}^{\infty} \frac{1}{n}$.
   (c) $\sum_{n=1}^{\infty} (4^n - 1)^n$ when $p = 2$
   (d) $\sum_{n=1}^{\infty} (4^n - 1)^n$ when $p = 3$.
5. Show that every convergent sequence in $\mathbb{Q}_p$ is Cauchy. *[Hint: the same is true in any metric space.]*
6. Let $(\alpha_n)_n$ be a Cauchy sequence in $\mathbb{Q}_p$, and let $\alpha_n = \sum_{k=v_p(\alpha_n)}^{\infty} a_{n,k} p^k$ be the p-adic expansion. Define $x_n := \sum_{k=v_p(\alpha_n)}^{n} a_{n,k} p^k$ to be the sum of the terms up to the nth digit in the expansion of α_n.

   Show that $(x_n)_n$ is a Cauchy sequence.

Section C

7. (*p-Adic Ratio Tests*) Let $(\alpha_n)_n$ be a sequence of elements in $\mathbb{Q}_p$, and assume that the limit

$$\lambda := \lim_{n\to\infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right|_p$$

exists.

(a) Show that the series $\sum_{n=1}^{\infty} \alpha_n$ converges if $\lambda < 1$, and diverges if $\lambda > 1$.

(b) Assume that $\lambda \neq 0$. Let $x \in \mathbb{Q}_p$, and consider the power series $\sum_{n=1}^{\infty} \alpha_n x^n$. Deduce that this converges if $|x|_p < 1/\lambda$ and diverges if $|x|_p > 1/\lambda$.

8. Using the previous question, or otherwise, determine for which $X \in \mathbb{Q}_p$ the following series converge.

(a) $\sum_{n=1}^{\infty} X^n$;

(b) $\sum_{n=1}^{\infty} \frac{X^n}{p^n}$;

(c) $\sum_{n=1}^{\infty} p^n X^n$;

(d) $\sum_{n=1}^{\infty} (pn)! X^n$