Introduction to the p-adic numbers
Exercise Sheet 1

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This exercise sheet is split into three sections:

- **A** extremely concrete computations to help unpack definitions;
- **B** theoretical questions which use only major results/definitions in the course;
- **C** theoretical results requiring some thought.

The recommended approach is to focus primarily on sections B and C once you are comfortable. You should only answer questions in section A where you're not confident with the definitions of the objects involved.

Section A

1. Compute the following
   (a) $v_3(-54)$
   (b) $v_5(0.0625)$
   (c) $v_2(4!)$
2. Compute the following for every prime number p
   (a) $|12|_p$
   (b) $\left|\frac{753}{20}\right|_p$
   (c) $\left|\frac{23}{18}\right|_p$
3. Compute the 2-adic expansion of x for
   (a) $x = 617$
   (b) $x = \frac{89}{4}$
4. Determine the first 3 non-zero digits in the 5-adic expansion of x for
   (a) $x = 1/3$
   (b) $x = -1/4$
   (c) $x = 3/2$
5. Which of the following are Cauchy sequences with respect to $|\cdot|_2$?
   (a) $a_n := 2^n$
   (b) $a_n := \sum_{k=0}^{n} 2^k$

Section B

6. Let $x \in \mathbb{Q}$ be a rational number with p-adic expansion $\sum_{k=v}^{\infty} a_k p^k$.
   (a) Determine the p-adic expansion of $-x$.

(b) If $x \in \mathbb{Z}$, then show that the digits in the p-adic expansion of x are eventually all $(p-1)$ or eventually all 0.

(c) Determine the p-adic expansion of $\frac{1}{1-p}$.

7. Let n be a positive integer, and let $n = a_t p^t + \cdots + a_1 p + a_0$ be its p-adic expansion. We will determine the valuation of the factorial $n!$.

   (a) Show that
   $$v_p(n!) = \sum_{k=1}^{t} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

   (b) Let $s := a_0 + a_1 + \cdots + a_t$ be the sum of the p-adic digits of n. Show that
   $$v_p(n!) = \frac{n-s}{p-1}.$$

   (c) Use this to determine $|n!|_p$ in terms of the p-adic expansion of n.

8. In this question we show that p-adic triangles have to be isosceles. Recall that a triangle is isosceles if two of its sides are of the same length.

   Let $x, y, z \in \mathbb{Q}$ be the vertices of our triangle. Assume that one side length is different to the other two, i.e. both $|x-y|_p \neq |x-z|_p$ and $|x-y|_p \neq |y-z|_p$. Show that the other two side lengths must be the same, i.e. $|y-z|_p = |x-z|_p$.

9. Let $x \in \mathbb{Q}$, we will investigate some identities which use 'all p' simultaneously.

   (a) Show that if $x \neq 0$ then
   $$\|x\| \prod_p |x|_p = 1.$$

   (b) Show that $x \in \mathbb{Z}$ if and only if $|x|_p \leq 1$ for every prime number p.

10. Let $N \geq 2$ be an integer which is *not* a prime. For $x, y \in \mathbb{Z} \setminus \{0\}$, define
   $$v_N(x) := \max\{n \in \mathbb{Z} \ : \ N^n \mid x\}, \qquad v_N\left(\frac{x}{y}\right) = v_N(x) - v_N(y),$$
   and set $v_N(0) := \infty$. If $|\cdot|_N = N^{-v_N(\cdot)}$, then:

   - is this well defined? What conditions should we add if not?
   - when this is well defined, what properties from Proposition 1.9 hold or fail for $|\cdot|_N$?

   Give proofs or counterexamples!

Section C

11. In this exercise we will prove that the p-adic expansion of a rational number is eventually periodic (Lemma 1.21 in the notes). *You may wish to make use of your answer to Question 6 here!* Let $x \in \mathbb{Q}$, and write $x = \lim\limits_{N\to\infty}^{(p)} \sum_{k=v}^{N} a_k p^k$ for the p-adic expansion.

   (a) Show that the p-adic expansion of x is eventually periodic if and only if the same is true for $-x$. Hence we can reduce to the case $x < 0$.

   (b) Explain why the p-adic expansion of x is eventually periodic if and only if the same is true for $p^k x$ for every $k \in \mathbb{Z}$. Hence can assume that $|x|_p = 1$.

   (c) Show that if x and y have eventually periodic p-adic expansions then so does $x + y$.

(d) Using the previous parts and Question 6(b), show that we may reduce to the case $-1 < x < 0$ and $|x|_p = 1$.

(e) We now prove the reduced case. Write $x = \frac{a}{b}$ for a, b coprime to p, $a < 0$, $b > 1$.

i. Show that there exists a positive integer k such that $p^k \equiv 1 \mod b$.

ii. Rewrite x in such a way that its denominator is $1 - p^k$.

iii. Show that the p-adic expansion of x is periodic.

12. Let $x \in \mathbb{Q}$. Consider the sequence $(e_n)_{n \geq 0}$ where

$$e_n := \sum_{i=0}^{n} \frac{x^i}{i!}.$$

Show that $(e_n)_{n \geq 0}$ is a Cauchy sequence with respect to $|\cdot|_p$ in each of the following regimes

(a) if $p > 2$ and $|x|_p < 1$;

(b) if $p = 2$ and $|x|_p < 1/2$.