JACOBIANS & MODELS PART II

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Disclaimer. These notes were taken live during lectures at the Introduction to SAGA winter school held at the CIRM from 30th January to 3rd February 2023. Any errors are the fault of the transcriber and not of the lecturer.

Lecture 1

There is a whole formalism where starting with a variety $V/\mathbb{Q}$, one obtains Galois representations and L-functions. Many conjectures are concerned with this: BSD, Langlands, Riemann hypothesis, etc..

You might think this is interesting, but then you see the definitions

$$L(H^i(V), s) = \prod_p \frac{1}{F_p(p^{-s})},$$

where $F_p(T) := \det\left(1 - \mathrm{Frob}_p^{-1} T \mid \mathrm{H}^i_{\text{ét}}(V_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)^{I_p}\right)$. At this point maybe you decide to go do something else with your life...

Nonetheless we shall aim to explain this, especially for H^1 of curves.

1. Semistable Curves over $\mathbb{F}_p$

Let $V/\mathbb{F}_p$ be a projective variety, with $n = \dim V$. Let $\ell \neq p$ be a prime and consider the étale cohomology groups

$$H^i(V) := \mathrm{H}^i_{\text{ét}}(V_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$$

for $0 \leq i \leq 2n$. These are ℓ-adic representations of $G_{\mathbb{F}_p}$, which is isomorphic to $\widehat{\mathbb{Z}}$ with canonical topological generator given by Frobenius $\mathrm{Frob} : x \mapsto x^p$. Moreover, since the representation is continuous, the action of frobenius completely determines the representation: in other words, only one matrix matters!

If semisimple (known for curves and abelian varieties), then this is determined completely by the characteristic polynomial

$$L_i(T) := \det\left(1 - \mathrm{Frob}^{-1} T \mid H^i(V)\right)$$

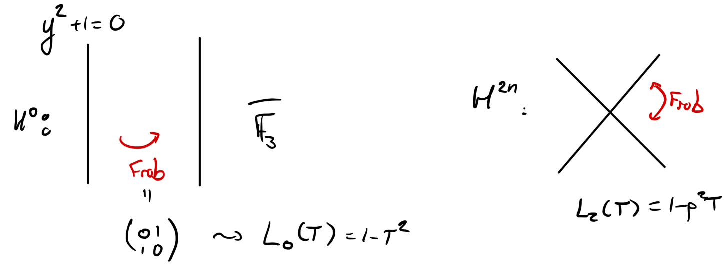
Facts:

- $H^0(V) = \mathbb{Q}_\ell V_1 \oplus \cdots \oplus \mathbb{Q}_\ell V_n$, where V_i are the connected components of $V_{\overline{\mathbb{F}_p}}$. In particular, if $V_{\overline{\mathbb{F}_p}}$ is connected then $H^0(V) \cong \mathbb{Q}_\ell$ and $L_0(T) = 1 - T$.
- $H^{2n}(V) = \mathbb{Q}_\ell(n) I_1 \oplus \cdots \oplus \mathbb{Q}_\ell(n) I_k$ where I_j are the irreducible components of $V_{\mathbb{F}_p}$ and $\mathbb{Q}_\ell(n)$ is the 1-dimensional representation with Frob acting as p^{-n}. In particular if $V_{\overline{\mathbb{F}_p}}$ is irreducible then $H^{2n}(V) \cong \mathbb{Q}_\ell(n)$ and so $L_{2n}(T) = 1 - p^n T$

- If $V = C$ is a smooth geometrically irreducible curve, then
$$H^1(C) = (V_\ell \mathrm{Jac}(C))^\vee,$$
is the dual of the Tate module.
- There is a zeta function
$$Z(T) := \exp\left(\sum_{k\geq 1} \frac{V(\mathbb{F}_{p^k})}{k} T^k\right) = \frac{L_1(T)L_3(T)\dots L_{2n-1}(T)}{L_0(T)L_2(T)\dots L_2n(T)}.$$
- If V is smooth then $L_i(T) = \prod_j (1-\alpha_j^{(i)}T)$ for some $\alpha_j^{(i)} \in \mathbb{C}$ with $\left|\alpha_j^{(i)}\right| = p^{i/2}$

The final two are consequences of the Weil conjectures, and in particular we can compute all the $L_i(T)$ from knowing $\#V(\mathbb{F}_{p^k})$for all the $k \geq 1$.

Example 1. *$y^2+1=0$ over $\mathbb{F}_3$ decomposes as two lines which are swapped by Frobenius and so* Frob *acts as* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *on $H^0(V)$, and has characteristic polynomial $L_0(T) = 1 - T^2$.*



Similarly, looking at the action on the lines above we note that $H^2(V)$ has characteristic polynomial $L_2(T) = 1 - p^2T^2$.

Example 2. *Consider $C : y^2 = x^2(x+1)/\mathbb{F}_p$. Then this is a split nodal cubic over $\mathbb{F}_p$, and*
$$\#C(\mathbb{F}_p^k) = \{(0,0)\} \cup \mathbb{F}_{p^k}^\times.$$
In particular $\#C(\mathbb{F}_{p^k}) = p^k$. Thus
$$Z(T) = \exp\left(\sum_{k\geq 1} \frac{p^k}{k} T^k\right) = \exp\left(-\log(1-pT)\right) = \frac{1}{1-pT} = \frac{L_1(T)}{L_0(T)L_2(T)}.$$
We compute that $L_0(T) = 1 - T$ since the curve is connected, and that $L_2(T) = 1 - pT$ since the curve is irreducible, and so via the Weil conjectures
$$Z(T) = \frac{1-T}{(1-T)(1-pT)},$$
so $L_1(T) = 1 - T$. This then tells us that our $H^1(C)$ is the one-dimensional trivial representation
$$\mathrm{H}^1_{\text{ét}}(C_{\mathbb{F}_p}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.$$

Example 3. *Consider* $y^2 = x^2(x+\eta)$ *where* $\eta \in \mathbb{F}_p^\times \backslash \mathbb{F}_p^{\times 2}$ *then we have a nonsplit nodal cubic. Then*

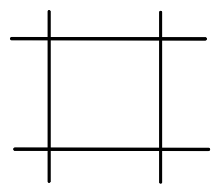
$$\#C(\mathbb{F}_{p^k}) = p^k + 1 - (-1)^k,$$

and so we compute

$$Z(T) = \frac{1+T}{(1-T)(1-pT)}$$

so $L_1(T) = 1 + T$*. We leave it as an exercise to write down what the Galois action on cohomology is.*

Example 4. *C four copies of* $\mathbb{P}^1$ *arranged in a square, say obtained from reducing a regular model of a type* I_4 *elliptic curve over* $\mathbb{Q}_p$*.*



For example: $y^2 = x^3 + x^2 + p^4$*. Then we count*

$$\#C(\mathbb{F}_{p^k}) = 4(p^k + 1) - 4 = 4p^k,$$

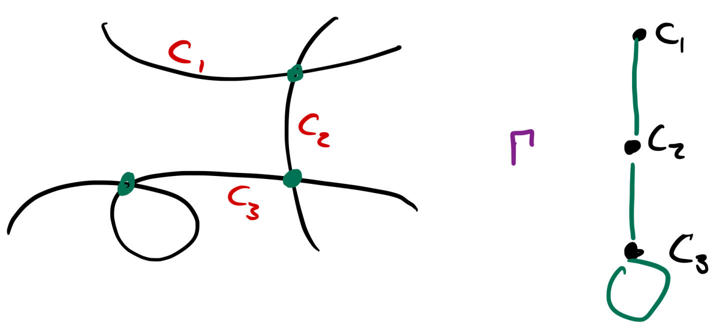
and $L_0(T) = 1 - T$*,* $L_2(T) = (1 - pT)^4$*. Computing Zeta we get*

$$Z(T) = \exp(-4\log(1-pT)) = \frac{1}{(1-pT)^4} = \frac{1-T}{(1-T)(1-pT)^4},$$

and so $L_1(T) = 1 - T$*.*

In general if $C/\mathbb{F}_p$ is a *semistable* curve (i.e. only ordinary double points as singularities) then write $\tilde{C_1}, \ldots, \tilde{C_m}$ for the normalisations of the irreducible components of $C/\overline{\mathbb{F}_p}$.

Definition 5. The dual group Γ is then the graph with m vertices, labelled by the components, and edges corresponding to each double point. We view this as a topological space.



Then it is not hard to see that

Theorem 6. *We have a decomposition into an abelian part and a toric part:*

$$H^1(C) = H^1_{\text{ab}}(C) \oplus H^1_t(C),$$

where

$$H^1_{\text{ab}}(C) = H^1(\tilde{C_1}) \oplus \cdots \oplus H^1(\tilde{C_m}),$$

and

$$H^1_t(C) = H^1_{\text{top}}(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Note that both the abelian and the toric part carry an action of $G_{\mathbb{F}_p}$, moreover we get that the eigenvalues on the abelian part have absolute value $p^{1/2}$ and on the toric part they have absolute value 1.

Remark 7. The dimension of the toric part is the number of loops in Γ.

Lecture 2

2. Semistable curves over $\mathbb{Q}_p$

Recall the exact sequence

$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1$$

where I_p is the inertia subgroup, and we fix a lift Frob $\in G_{\mathbb{Q}_p}$ of the map $x \mapsto x^p$ in $G_{\mathbb{F}_p}$.

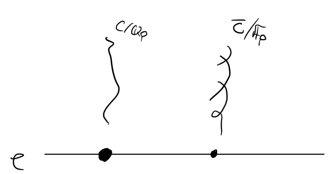
Let $C/\mathbb{Q}_p$ be a nice (smooth projective absolutely irreducible) genus g curve over $\mathbb{Q}_p$. Our goal will be to completely describe the action of Galois on $2g$-dimensional $\mathbb{Q}_\ell$-vector space

$$H^1(C) := \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell) = (V_\ell \mathrm{Jac}(C))^\vee$$

and the characteristic polynomial

$$F(T) = \det(1 - \mathrm{Frob}^{-1} \mid H^1(C)^{I_p}),$$

in terms of the geometry of a regular model $\mathscr{C}/\mathbb{Z}_p$ and its special fibre $\overline{C}/\mathbb{F}_p$.



Example 8. *If C has good reduction, meaning $\overline{C}$ is smooth, then I_p acts trivially on $H^1(C)$ and*

$$H^1(C) \cong H^1(\overline{C})$$

as $G_{\mathbb{F}_p}$-reps, and

$$F(T) = L_1(\overline{C}, T).$$

For elliptic curves this is Ogg–Shafarevich, in general it is Serre–Tate.

Example 9 (Tate curve)**.** *If $C = E$ is an elliptic curve with split multiplicative redution then recall that there is an isomorphism (as $G_{\mathbb{Q}_p}$-module)*

$$E(\overline{\mathbb{Q}_p}) \cong \overline{\mathbb{Q}_p}^{\times}/q^{\mathbb{Z}}$$

for some $q \in p\mathbb{Z}_p$. Note that clearly

$$E[\ell^n] \cong \langle \zeta_{\ell^n},\ \sqrt[\ell^n]{q} \rangle\,.$$

This shows that the action of $G_{\mathbb{Q}_p}$ on the Tate module $T_\ell E$ is given by

$$\begin{pmatrix} \chi_\ell & \tau_\ell \\ 0 & 1 \end{pmatrix}$$

where χ_ℓ is the ℓ-adic cyclotomic character and τ_ℓ is the tame chracter. Thus on $H^1(E)$ the action is given by

$$\begin{pmatrix} \chi_\ell^{-1} & 0 \\ \tau_\ell & 1 \end{pmatrix} =: \mathrm{Sp}_2.$$

Thus it is clear that $H^1(E)^{I_p} = \mathbb{Q}_\ell$ and $F(T) = 1 - T$.

In general we have the following theorem of Grothendieck for semistable curves.

Theorem 10 (Grothendieck)**.** *Let $C/\mathbb{Q}_p$ be a nice curve with demistable reduction. Then*

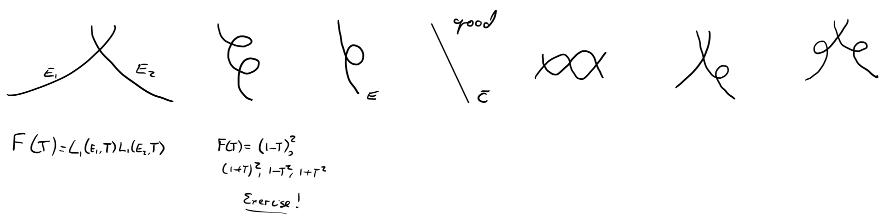
$$H^1(C)^{I_p} \cong H^1(\overline{C}),$$

and

$$H^1(C) \cong H^1_{\mathrm{ab}}(\overline{C}) \oplus \left(H^1_t(\overline{C}) \otimes \mathrm{Sp}_2\right)$$

where $H^1_t(\overline{C}) \otimes \mathrm{Sp}_2 = H^1_{top}(\Gamma, \mathbb{Z}) \otimes \mathrm{Sp}_2$ for Γ the dual graph of $\overline{C}$.

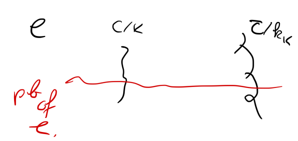
Example 11. *If C is of genus 2, then there are 7 semistable types*



3. General Curves

Notation 12. We will take

- $C/\mathbb{Q}_p$ a nice (smooth proj. geom. irr.) curve;
- $K/\mathbb{Q}_p$ a finite Galois extension for which C/K has semistable reduction;
- $\mathscr{C}/\mathcal{O}_K$ the minimal regular model over $\mathcal{O}_K$;
- $\overline{C}$ the special fibre;
- Γ the dual graph of $\overline{C}$.



Then we have maps

$$\begin{array}{ccc} & & C(K_{\mathrm{nr}}) \\ & \overset{\eta}{\nearrow} & \\ \mathscr{C}(\mathcal{O}_{K_{\mathrm{nr}}}) & & \\ & \underset{\mathrm{red}}{\searrow} & \\ & & \overline{C}(\overline{\mathbb{F}_p}) \end{array}$$

Theorem 13 (Dokchitser–Dokchitser–Morgan)**.** *There is a semilinear action $G_{\mathbb{Q}_p}$ on $\overline{C}(\overline{\mathbb{F}_p})$ given on non-singular points by, for $\sigma \in G_{\mathbb{Q}_p}$*

$$\overline{C}(\overline{\mathbb{F}_p}) \xrightarrow{\mathrm{red}^{-1}} \mathscr{C}(\mathcal{O}_{K_{\mathrm{nr}}}) \xrightarrow{\eta} C(K_{\mathrm{nr}}) \xrightarrow{\sigma} C(K_{\mathrm{nr}}) \xrightarrow{\eta^{-1}} \mathscr{C}(\mathcal{O}_{K_{\mathrm{nr}}}) \longrightarrow \overline{C}(\overline{\mathbb{F}_p})$$

It induces actions on the dual graph Γ, $H^1_t(\overline{C})$, $H^1_{\mathrm{ab}}(\overline{C})$ and

$$H^1(C) \cong H^1_{\mathrm{ab}}(\overline{C}) \oplus H^1_t(\overline{C}) \otimes \mathrm{Sp}_2$$

as $G_{\mathbb{Q}_p}$-modules

Example 14. *$C : y^2 = x^3 + p^4/\mathbb{Q}_p$ for $p > 3$. This has type IV additive reduction. $K = \mathbb{Q}_p(\sqrt[3]{p})$, write $\pi = \sqrt[3]{p}$. Take a substitution: $X = x/p\pi$, $Y = y/p^2$ then*

$$\begin{array}{ccc} C/K : y^2 = x^3 + p^4 & \underset{\eta}{\longleftarrow} & \mathscr{C}/\mathcal{O}_K : Y^2 = X^3 + 1 \\ & & \big\downarrow \mathrm{red} \\ & & \overline{C}/\mathbb{F}_p : Y^3 = X^3 + 1 \end{array}$$

For example, $\sigma \in I_p$ acts as

$$\begin{array}{l} \overline{C}(\overline{\mathbb{F}_p}) \ni (X, Y) \xrightarrow{\mathrm{lift}} (\tilde{X}, \tilde{Y}) \xrightarrow{\eta} (\pi p \tilde{X}, p^2 \tilde{Y}) \\ \xrightarrow{\sigma} (\sigma(\pi) p \tilde{X}, p^2 \tilde{Y}) \xrightarrow{\eta^{-1}} (\tfrac{\sigma(\pi)}{\pi} \tilde{X}, \tilde{Y}) \longrightarrow (\tfrac{\sigma(\pi)}{\pi} x, y) \end{array}$$

Let $\psi : I_{\mathbb{Q}_p} \to \mu_3$ be the character $\sigma \mapsto \sigma(\pi)/\pi$, then we find that I_p acts on $H^1(C)$ as

$$\begin{pmatrix} \psi & 0 \\ 0 & \psi^{-1} \end{pmatrix}.$$

In particular, $H^1(C)^{I_p} = 0$ and $F(T) = 1$.

In fact $F(T) = 1$ for every elliptic curve with additive reduction.

Question 15. *How do we find K and $\mathscr{C}$ in practice? E.g. for hyperelliptic curves when $p = 2$?*