

# GRUNWALD PROBLEM AND SOLVABLE DESCENT

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## LECTURE 1

Let  $K$  be a number field, write  $X(K_\Omega) := \prod_v X(K_v)$ . By a *(RC) fibration* we will mean a smooth proper morphism with (rationally connected) geometrically integral fibres.

**Definition 1.** For a smooth variety  $Y/K$ , we set  $\text{Br}_{\text{un}}(Y) := \text{Br}(Y^c)$  for a(ny) smooth compactification  $Y^c$  of  $Y$ .

The theorem we will be focussing on is the following, which we will often refer to as the *fibration theorem*.

**Theorem 2** (Fibration theorem). *Let  $f : X \rightarrow Q$  be a rationally connected fibration, where  $Q = R_{E/K}\mathbb{G}_m$  is a quasi trivial torus. Assume that  $\exists T/\bar{K}$  with*

- $T \twoheadrightarrow \bar{Q} = Q \times_k \bar{K}$  epimorphism of tori;
- there is a  $T$ -action on  $\bar{X} = X \times_K \bar{K}$  such that  $f \times_K \bar{K}$  is  $T$ -equivariant.

Then

$$\overline{\bigcup_{q \in Q(K)} X_q(\mathbb{A}_K)^\bullet}^{\text{Br}(X_q)} = X(K_\Omega)^\bullet{}^{\text{Br}_{\text{un}}(X)}.$$

Here the  $\bullet$  is to denote that at the archimedean components we only have the connected components.

## MOTIVATION

The Grunwald problem is the following. Let  $G$  be a finite soluble  $K$ -group, then does

$$H^1(K, G) \rightarrow \prod_v H^1(K_v, G)$$

have dense image? Via the regular representation,  $G$  embeds in  $\text{SL}_n$ , so we have a short exact sequence of pointed schemes

$$1 \rightarrow G \rightarrow \text{SL}_n \rightarrow \text{SL}_n/G \rightarrow 1.$$

Taking Galois fixed points we have a commutative diagram

$$\begin{array}{ccc} (\text{SL}_n/G)(K) & \longrightarrow & H^1(K, G) \\ \downarrow & & \downarrow \\ \prod_v (\text{SL}_n/G)(K_v) & \longrightarrow & \prod_v H^1(K_v, G) \end{array}$$

By Hilbert 90 the horizontals are surjective and so we know that the vertical image of  $H^1(K, G)$  lands in the image of the Brauer–Manin set so defining

$$\prod_v H^1(K_v, G)^{\text{BM}} = \partial \left( \prod_v (\text{SL}_n/G)(K_v)^{\text{Br}_{\text{un}}}\right)$$

I will now explain how Harpaz–Wittenberg infer so-called “solvable descent” from the fibration theorem.

#### DESCENT TYPES

**Definition 3.** Let  $X/K$  be a geometrically integral variety. A finite descent type (FDT) is a finite cover  $Y \rightarrow X_{\overline{K}}$  which is irreducible and such that  $\overline{K}(Y)/\overline{K}(X)$  is Galois. We say it is solvable, abelian, etc. if the Galois group is.

**Definition 4.** Let  $\overline{Y} \rightarrow X_{\overline{K}}$  be a finite descent type. A torsor of type  $\overline{Y}$  is a cover  $Y \rightarrow X$  such that  $Y \times_K \overline{K} = \overline{Y}$  as  $X_{\overline{K}}$ -schemes.

Then we have

**Theorem 5** (Harpaz–Wittenberg). *Assume  $X$  is smooth, geometrically integral, and  $\overline{Y}/X_{\overline{K}}$  is a solvable FDT. Then if the fibration theorem holds,*

$$\bigcup_{\substack{f: Y \rightarrow X \\ \text{type } \overline{Y}}} \overline{f(Y(K_{\Omega})^{\text{Br}_{\text{un}}(Y)})} = X(K_{\Omega})^{\text{Br}_{\text{un}}(X)}$$

**Corollary 6.**

$$\overline{(\text{SL}_n/G)(K)} = (\text{SL}_n/G)(K_{\Omega})^{\text{Br}_{\text{un}}(\text{SL}_n/G)}$$

*Proof.* Take  $\overline{Y} = \text{SL}_{n, \overline{K}}$ . Then all torsors of type  $\overline{Y}$  are  $\text{SL}_{n, K}$  by Hilbert 90, so applying Harpaz–Wittenberg with  $X = \text{SL}_n/G$  and writing  $\pi : \text{SL}_n \rightarrow \text{SL}_n/G$  we have

$$\overline{\pi(\text{SL}_n(K_{\Omega})^{\text{Br}_{\text{ur}}(\text{SL}_n)})} = (\text{SL}_n/G)(K_{\Omega})^{\text{Br}_{\text{ur}}(\text{SL}_n/G)}.$$

It is known that  $\text{SL}_n(K)$  is dense in  $\text{SL}_n(K_{\Omega})^{\text{Br}_{\text{ur}}(\text{SL}_n)}$ , and since the map  $\pi$  is a closed map we must thus have

$$\overline{\pi(\text{SL}_n(K))} = (\text{SL}_n/G)(K_{\Omega})^{\text{Br}_{\text{ur}}(\text{SL}_n/G)}.$$

Since  $\pi(\text{SL}_n(K)) \subseteq (\text{SL}_n/G)(K)$ , we have the claim.  $\square$

*Idea of Proof.* Roughly:

- One reduces to the abelian case by induction.
- Without loss of generality,  $X(K_{\Omega})^{\text{Br}_{\text{un}}(X)} \neq \emptyset$  so that via Poitou–Tate  $(\overline{Y} \rightarrow X_{\overline{K}}, \overline{G})$  can be descended to a pair  $(Y \rightarrow X, G/K)$  with  $G$  abelian.
- There is a short exact sequence (see Lemma 33)

$$1 \rightarrow G \rightarrow T \rightarrow Q \rightarrow 1$$

where  $T$  is a  $K$ -torus, and  $Q$  is quasitrivial. Let  $Z = (Y \times T)/G$

$$\begin{array}{ccc} Z & \xrightarrow{f_0} & T/G = Q \\ \downarrow \pi & & \\ Y/G = X & & \end{array}$$

Start with a  $K_\Omega$ -point downstairs  $(x_v)_v$  on  $X$ , approximate it with  $(\tilde{x}_v)_v$  and then lift it using open descent theory under tori to a point  $(z_v)_v \in Z_\sigma(K_\Omega)$  for some  $\sigma \in H^1(K, T)$ . That is, we are using that

$$\overline{\bigcup_{\sigma \in H^1(K, T)} \pi_\sigma (Z_\sigma(K_\Omega)^{\text{Brun}(Z_\sigma)})} = X(K_\Omega)^{\text{Brun}(X)}$$

We may now apply the fibration theorem to a relative compactification of  $f_0$ . Then we can approximate  $(z_v)_v \in Z(K_\Omega)^{\text{Brun}(Z)}$  with an adelic point on a fiber  $f_0^{-1}(q)$  for some  $q \in Q(K)$  which is unobstructed. This is basically the argument.  $\square$

### HORIZONTAL BRAUER GROUP

We take a quick digression on derived categories. Let  $\mathcal{A}$  be an abelian category with enough injectives, and let  $\mathcal{I} \subseteq \mathcal{A}$  be the full subcategory of injective objects. Then we have a category of (unbounded) chain complexes  $\text{Ch}(\mathcal{A})$  in  $\mathcal{A}$ . Quotienting this category by homotopy of chain complexes we get  $\mathcal{K}(\mathcal{A})$ , and then denote by

$$\mathcal{D}^+(\mathcal{A}) = \mathcal{K}^+(\mathcal{I}).$$

We write

$$\tau_{\geq 1}(\cdots \rightarrow A^{-1} \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots) = (\cdots \rightarrow 0 \rightarrow A^1/\text{im}(A^0) \rightarrow A^2 \rightarrow \cdots).$$

Now for the horizontal Brauer group. Let  $f : X \rightarrow Y$  be a fibration with  $Y$  regular and integral.

**Definition 7.** The horizontal Brauer group is

$$\text{Br}_h(X/Y) = H^2(Y, \tau_{\geq 1} Rf_* \mathbb{G}_m).$$

We elaborate slightly on this construction, for the reader whose derived categories are a little rusty. Take an injective resolution of étale sheaves on  $X$

$$C_\bullet : 0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{I}_0 \rightarrow \mathcal{I}_1 \rightarrow \mathcal{I}_2 \rightarrow \cdots$$

and one of étale sheaves on  $Y$

$$D_\bullet : 0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{J}_0 \rightarrow \mathcal{J}_1 \rightarrow \mathcal{J}_2 \rightarrow \cdots$$

Then build the Cartan–Eilenberg double complex from  $f_* C_\bullet$  and  $D_\bullet$ , so you have a double complex of sheaves on  $Y$

$$\begin{array}{ccccccc} \mathbb{G}_m & \longrightarrow & f_* \mathcal{I}_0 & \longrightarrow & f_* \mathcal{I}_1 & \longrightarrow & f_* \mathcal{I}_2 \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_0 & \longrightarrow & \mathcal{A}_{00} & \longrightarrow & \mathcal{A}_{01} & \longrightarrow & \mathcal{A}_{02} \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_1 & \longrightarrow & \mathcal{A}_{10} & \longrightarrow & \mathcal{A}_{11} & \longrightarrow & \mathcal{A}_{12} \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_2 & \longrightarrow & \mathcal{A}_{20} & \longrightarrow & \mathcal{A}_{21} & \longrightarrow & \mathcal{A}_{22} \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \ddots & & \ddots & & \ddots \end{array}$$

Truncate this double complex via  $\tau_{\geq 1}$ , here meaning that you are truncating columns, so (after excluding the usual rows/columns) you have the complex of sheaves on  $Y$

$$\begin{array}{ccccccccc}
\mathcal{A}_{01}/\mathrm{im}(\mathcal{A}_{00}) & \longrightarrow & \mathcal{A}_{02} & \longrightarrow & \mathcal{A}_{03} & \longrightarrow & \mathcal{A}_{04} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{A}_{11}/\mathrm{im}(\mathcal{A}_{10}) & \longrightarrow & \mathcal{A}_{12} & \longrightarrow & \mathcal{A}_{13} & \longrightarrow & \mathcal{A}_{14} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{A}_{21}/\mathrm{im}(\mathcal{A}_{20}) & \longrightarrow & \mathcal{A}_{22} & \longrightarrow & \mathcal{A}_{23} & \longrightarrow & \mathcal{A}_{24} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}$$

By  $H^2(Y, \tau_{\geq 1} Rf_* \mathbb{G}_m)$  we mean take global sections and then the second total cohomology (i.e. the second homology of the diagonal complex). Now, what do we know about this horizontal Brauer group?

**Theorem 8.** *The following holds.*

(1) *there exists a natural exact sequence*

$$\mathrm{Br}(Y) \xrightarrow{f^*} \mathrm{Br}(X) \rightarrow \mathrm{Br}_h(X/Y) \xrightarrow{\partial} H^3(Y, \mathbb{G}_m) \xrightarrow{f^*} (X, \mathbb{G}_m).$$

(2) *If RC:  $\mathrm{Br}_h(X/Y) \cong \mathrm{Br}_h(X_y/y)$  for all  $y \in H \subseteq Y$  Hilbertian.*

(3) *If RC: If  $Y$  is a variety over  $K$  a number field (so  $H^3(K, \mathbb{G}_m) = 0$ ), then  $\mathrm{Br}_h(X_y/y) = \mathrm{Br}(X_y)/\mathrm{im}(\mathrm{Br}(K))$  for all  $y \in Y(K)$ .*

*Proof.* (1) We have an exact triangle

$$\mathbb{G}_m = f_* \mathbb{G}_m \rightarrow Rf_* \mathbb{G}_m \rightarrow \tau_{\geq 1} Rf_* \mathbb{G}_m \xrightarrow{+1}$$

applying  $H^*(Y, -)$  we get

$$\mathrm{Br}(Y) \xrightarrow{f^*} H^2(Y, Rf_* \mathbb{G}_m) = \mathrm{Br}(X) \rightarrow \mathrm{Br}_h(X/Y) \xrightarrow{\partial} H^3(Y, \mathbb{G}_m) \xrightarrow{f^*} H^3(X, \mathbb{G}_m).$$

(2) We then take on  $X_{\mathrm{et}}$  the exact sequence

$$1 \rightarrow \mu_\infty \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow 1$$

inducing an exact triangle on  $Y_{\mathrm{et}}$

$$\tau_{\geq 1} Rf_* \mu_\infty \rightarrow \tau_{\geq 1} Rf_* \mathbb{G}_m \rightarrow \tau_{\geq 1} Rf_* (\mathbb{G}_m \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{+1}.$$

Now since  $f$  is an RC-fibration, the fibres are simply connected and so the homology in degree 1 of the first complex satisfies  $R^1 f_* \mu_\infty = 0$ . The second on the right is  $R^2 f_* (\mathbb{G}_m \otimes \mathbb{Q}) = R^2 f_* (\mathbb{G}_m) \otimes \mathbb{Q} = 0$  since before tensoring this is torsion. Now looking at cohomology in low degrees we can truncate in degrees 1 and 2 to get

$$\tau_{[2]} Rf_* \mu_\infty \rightarrow \tau_{[1,2]} Rf_* \mathbb{G}_m \rightarrow \tau_{[1]} Rf_* (\mathbb{G}_m \otimes \mathbb{Q}) \xrightarrow{+1}.$$

We now rotate the triangle to get

$$\tau_{[1]} Rf_* (\mathbb{G}_m \otimes \mathbb{Q})[1] \rightarrow \tau_{[2]} Rf_* \mu_\infty \rightarrow \tau_{[1,2]} Rf_* \mathbb{G}_m \xrightarrow{+1}$$

so replacing with homology we have

$$R^1 f_* (\mathbb{G}_m \otimes \mathbb{Q})[2] \rightarrow R^2 f_* \mu_\infty[2] \rightarrow \tau_{[1,2]} \mathbb{G}_m \xrightarrow{+1}.$$

Okay, so we're now going to take the hypercohomology for this exact sequence. We get

$$H^1(Y, \tau_{[1,2]} Rf_* \mathbb{G}_m) \rightarrow H^0(Y, R^1 f_* \mathbb{G}_m \otimes \mathbb{Q}) \rightarrow H^0(Y, R^2 f_* \mu_\infty) \rightarrow \mathrm{Br}_h(X/Y) \rightarrow H^1(Y, R^1 f_* \mathbb{G}_m \otimes \mathbb{Q}).$$

Here  $R^1 f_* \mathbb{G}_m = \mathrm{NS}_{Y/X}$ . Let  $Y'/Y$  be an étale cover trivialising the Neron-Severi scheme, then

$$H^1(\mathrm{Gal}(Y'/Y), R^1 f_* \mathbb{G}_m(Y') \otimes \mathbb{Q}) = 0$$

and so we can erase the last term above and get

$$H^1(Y, \tau_{[1,2]} Rf_* \mathbb{G}_m) \rightarrow H^0(Y, R^1 f_* \mathbb{G}_m \otimes \mathbb{Q}) \rightarrow H^0(Y, R^2 f_* \mu_\infty) \rightarrow \mathrm{Br}_h(X/Y) \rightarrow 0.$$

We have

$$\begin{array}{ccccc} \mathrm{NS}(X_{\bar{y}})^{\pi_1(Y, \bar{y})} & \longrightarrow & H^2(X_{\bar{y}}, \mu_\infty)^{\pi_1(Y, \bar{y})} & \longrightarrow & \mathrm{Br}_h(X/Y) \rightarrow 0 \\ \downarrow = & & \downarrow = & & \\ \mathrm{NS}(X_{\bar{y}})^{\mathrm{Gal}(Y'/Y)} & \longrightarrow & H^2(X_{\bar{y}}, \mu_\infty)^{\mathrm{Gal}(Y'/Y)} & \longrightarrow & \mathrm{Br}_h(X/Y) \rightarrow 0. \end{array}$$

Finally, (1)  $\implies$  (3) since  $H^3(K, \mathbb{G}_m) = 0$ .  $\square$

#### FILTRATION ON HORIZONTAL BRAUER GROUP

Let  $f : X \rightarrow Q$  be an (RC)-fibration with  $Q$  a quasitrivial torus. Define

- $H_{\mathrm{alg}}^3(Q, \mathbb{G}_m) := \ker(H^3(Q, \mathbb{G}_m) \rightarrow H^3(\bar{Q}, \mathbb{G}_m))$
- maps  $r, n$  by

$$\begin{array}{ccccc} \mathrm{Br} X & \longrightarrow & \mathrm{Br}_h(X/Q) & \xrightarrow{\partial} & H^3(Q, \mathbb{G}_m) \\ \downarrow n & \nearrow r & \downarrow = & & \\ \mathrm{Br} X_\eta & \longrightarrow & \mathrm{Br}_h(X_\eta/\eta) & \xrightarrow{\partial} & H^3(\eta, \mathbb{G}_m) \end{array}$$

here the equality follows from (2) of the previous theorem, since every open contains such a point  $y$ . Hence we get the existence of the map  $r$ .

We now define a filtration as follows.

- $F_0 \mathrm{Br}_h := r(\mathrm{Br} X)$
- $F_1 \mathrm{Br}_h := r(\mathrm{Br} X_\eta)$
- $F_2 \mathrm{Br}_h := \partial^{-1} H_{\mathrm{alg}}^3(Q, \mathbb{G}_m)$
- $F_3 \mathrm{Br}_h := \mathrm{Br}_h$

It is not yet clear that this is a filtration! We need the following.

**Lemma 9.**  $F_1 \mathrm{Br}_h \subseteq F_2 \mathrm{Br}_h$ .

*Proof.*

$$\begin{aligned} \partial(F_1) &\subseteq \ker(H^3(Q, \mathbb{G}_m) \rightarrow H^3(\eta, \mathbb{G}_m)) \\ &\subseteq \ker(H^3(Q, \mathbb{G}_m) \rightarrow H^3(\bar{\eta}, \mathbb{G}_m)) \\ &= \ker(H^3(Q, \mathbb{G}_m) \rightarrow H^3(\bar{Q}, \mathbb{G}_m)) \end{aligned}$$

where  $\bar{\eta} = \eta \times_K \bar{K}$ . Here we have used injectivity of the second map in

$$\begin{array}{ccccc} H^3(Q, \mathbb{G}_m) & \longrightarrow & H^3(\bar{Q}, \mathbb{G}_m) & \longrightarrow & H^3(\bar{\eta}, \mathbb{G}_m) \\ & & \downarrow = & & \downarrow = \\ & & H^3(\bar{K}[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \mathbb{G}_m) & \longrightarrow & H^3(\bar{K}(t_1, \dots, t_n), \mathbb{G}_m) & \longrightarrow & H^3(\bar{K}((t_1, \dots, t_n)), \mathbb{G}_m) \end{array}$$

which holds for tori: the composition of the bottom two maps is an isomorphism by Gille and Pianzola (2008).  $\square$

## LECTURE 2

Last time, we introduced a filtration on the horizontal Brauer group. Recall we had this diagram

$$\begin{array}{ccccc} \text{Br}X & \longrightarrow & \text{Br}_h(X/Q) & \xrightarrow{\partial} & H^3(Q, \mathbb{G}_m) \\ \downarrow n & \nearrow r & \downarrow = & & \\ \text{Br}X_\eta & \longrightarrow & \text{Br}_h(X_\eta/\eta) & \xrightarrow{\partial} & H^3(\eta, \mathbb{G}_m) \end{array}$$

This middle isomorphism is very general and holds when  $f : X \rightarrow Y$  is smooth proper with geometrically integral fibres. Recall that  $Q = \mathbb{R}_{E/K} Q$  is a quasitrivial torus and we can ensure that  $E/K$  is Galois with  $\Gamma = \text{Gal}(E/K)$  so that

$$Q = \text{Spec}(E[y_\sigma^{\pm 1}]_{\sigma \in \Gamma})^\Gamma.$$

Let  $y := y_{id}$ .

### REPRESENTATIVES FOR $F_1 \text{Br}_h$

We take the following notations.

- $\Gamma^* = \Gamma \setminus \{1\}$
- $\Gamma[2]^* = \{\tau \in \Gamma^* : \tau^2 = 1\}$
- $n = \max\{r : \mu_r \subseteq E\}$
- $\Gamma[2]** = \{\tau \in \Gamma[2]^* : \chi_{\text{cyc}}(\tau) \equiv \pm 1 \pmod{\gcd(8, n)}\}$ , note that this is the set of elements which act as complex conjugation or the identity on  $\mathbb{Q}(\zeta_8)$ .

Why should we be interested in this?

$$- \ker(H^1(K, \text{Br}\bar{Q}) \rightarrow H^1(K, \text{Br}\bar{\eta})) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus \Gamma[2]**}$$

$$- \text{For } \tau \in \Gamma[2]^* \text{ we have } H^1(\langle \tau \rangle, \mathbb{Z}/n\mathbb{Z}(1)) = \begin{cases} 0 & \text{if } \tau \notin \Gamma[2]** \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \tau \in \Gamma[2]** \end{cases}$$

- For  $\tau \in \Gamma[2]^*$ 
  - $Q \otimes_k E^\tau = \text{Spec}(E[y_\sigma^{\pm 1}]_{\sigma \in \Gamma}^{\langle \tau \rangle}) \supseteq \{y = y^\tau\} =: Z_\tau^0$ .
  - $Z_\tau \subseteq Q$  such that  $Z_\tau \otimes E = \bigcup_\sigma (Z_\tau^0 \otimes_{E^\tau} E)^\sigma$ . We have  $K(Z_\tau) = E^\tau(Z_\tau^0)$ .
  - $Q^\# := Q \setminus \bigcup_{\tau \in \Gamma[2]**} Z_\tau$ .

In the sequel, we shall frequently use the identification  $K(Z_\tau) = E^\tau(Z_\tau^0) = \text{Frac}(E^\tau[y_\sigma]_{\sigma \in \Gamma}/(y - y_\tau))$  to interpret  $y$  as an element of  $K(Z_\tau)$ .

If  $f : X \rightarrow Q$  is a fibration, with  $Q$  Galois quasitrivial as before, then write

$$Z'_\tau := f^{-1}(Z_\tau).$$

**Definition 10.** We define

$$\mathrm{Br}(X)_+ := \left\{ b \in \mathrm{Br}f^{-1}(Q^\#) : \begin{array}{l} \partial_{Z'_\tau} \in H^1(K(Z'_\tau), \mathbb{Q}/\mathbb{Z}) \\ \text{is split by } y^{1/n} \\ \forall \tau \in \Gamma[2]^{**} \end{array} \right\}.$$

Observe that an element of  $H^1(K(Z'_\tau), \mathbb{Q}/\mathbb{Z})$  is split by  $y^{1/n}$  if and only if it is split by  $y^{1/n_\tau}$  where  $n_\tau = \max\{d : \mu_d \subseteq E^\tau\}$ .

**Theorem 11.** *The map  $r$  induces a surjection  $\mathrm{Br}(X)_+ \twoheadrightarrow F_1\mathrm{Br}_h(X/Q)$ .*

#### PROVING THEOREM 11

$\mathrm{Br}(Q)_+ :=$  same way with  $Z'_\tau \mapsto Z_\tau$

$\mathrm{Br}(\overline{Q})_+ :=$  same way with  $\{Z'_\tau\}_{\tau \in \Gamma[2]^{**}} \mapsto \{(\overline{Z}_\tau^0)^\sigma\}_{\tau \in \Gamma[2]^{**}, \sigma \in \Gamma}$  and  $y \mapsto y^\sigma$

**Proposition 12.** *We have*

$$\ker(H^1(K, \mathrm{Br}(\overline{Q})) \rightarrow H^1(K, \mathrm{Br}(\overline{\eta}))) = \ker(H^1(K, \mathrm{Br}(\overline{Q})) \rightarrow H^1(K, (\mathrm{Br}(\overline{Q}))_+))$$

*Proof.* Consider the commutative diagram (the rows are not exact!!)

$$\begin{array}{ccccc} H^1(K, \mathrm{Br}(\overline{Q})) & \longrightarrow & H^1(K, \mathrm{Br}(\overline{Q})_+) & \longrightarrow & H^1(K, \mathrm{Br}(\overline{\eta})) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(E, \mathrm{Br}(\overline{Q})) & \longrightarrow & H^1(E, \mathrm{Br}(\overline{Q})_+) & \longrightarrow & H^1(E, \mathrm{Br}(\overline{\eta})) \end{array}$$

By Gille and Pianzola (2008), the composition of the bottom row is injective since  $\mathrm{Br}(\overline{\eta}) =_E \mathrm{Br}(\overline{Q}) \oplus M$ .

**Claim:** the second vertical above is also injective.

*Proof of claim:* Indeed, via inflation restriction it is enough to show that  $H^1(\Gamma, \mathrm{Br}(\overline{Q})_+^{\Gamma_E})$  vanishes. Let

$$\binom{\Gamma}{2}_{*(*)} := \left\{ \{\sigma, \tau\} \in \binom{\Gamma}{2} : \sigma\tau^{-1} \in \Gamma[2]^{*(*)} \right\}$$

Then

$$\mathrm{Br}(\overline{Q})_+ = \mathrm{Br}(\overline{Q}) \oplus \bigoplus_{\{\sigma, \tau\} \in \binom{\Gamma}{2}_{**}} (y^\sigma \cup (y^\tau - y^\sigma)) \cdot (\mathbb{Z}/n\mathbb{Z})(-1)$$

Taking  $\Gamma_E$  invariants, note that

$$\mathrm{Br}(\overline{Q}) = (\mathbb{Q}/\mathbb{Z})(-1)^{d(d-1)/2},$$

and so

$$\mathrm{Br}(\overline{Q})^{\Gamma_E} = \mathrm{Br}(\overline{Q})[n]$$

and hence

$$\mathrm{Br}(\overline{Q})_+^{\Gamma_E} = \bigoplus_{\{\sigma, \tau\} \in \binom{\Gamma}{2}} (y^\sigma \cup y^\tau) \cdot (\mathbb{Z}/n\mathbb{Z})(-1) \oplus \bigoplus_{\{\sigma, \tau\} \in \binom{\Gamma}{2}_{**}} (y^\sigma \cup (y^\tau - y^\sigma)) \cdot (\mathbb{Z}/n\mathbb{Z})(-1)$$

Note the identity

$$-y^\sigma \cup (y^\tau - y^\sigma) + y^\tau \cup (y^\sigma - y^\tau) = y^\sigma \cup y^\tau.$$

Doing some linear algebra with this we arrive at, for  $B_{\sigma,\tau} := (y^\sigma \cup y^\tau) \cdot (\mathbb{Z}/n\mathbb{Z})(-1)$  and  $C_{\sigma,\tau} := (y^\sigma \cup (y^\tau - y^\sigma)) \cdot (\mathbb{Z}/n\mathbb{Z})(-1)$ , the identity

$$\mathrm{Br}(\overline{Q})_+^{\Gamma E} = \bigoplus_{\{\sigma,\tau\} \in \binom{\Gamma}{2} \setminus \binom{\Gamma}{2}_{**}} B_{\sigma,\tau} \oplus \bigoplus_{\sigma\tau^{-1} \in \Gamma[2]**} C_{\sigma,\tau}.$$

Note that  $C_{\sigma,\tau}$  is an induced module from the trivial group, since the  $\Gamma$ -action permutes the basis freely, and so has trivial  $H^1$ , so we only care about the  $B_{\sigma,\tau}$  summand!

$$\begin{aligned} & \bigoplus_{\{\sigma,\tau\} \in \binom{\Gamma}{2} \setminus \binom{\Gamma}{2}_{**}} (y^\sigma \cup y^\tau) \cdot (\mathbb{Z}/n\mathbb{Z})(-1) \\ &= \bigoplus_{\{\sigma,\tau\} \in \binom{\Gamma}{2} \setminus \binom{\Gamma}{2}_*} (y^\sigma \cup y^\tau) \cdot (\mathbb{Z}/n\mathbb{Z})(-1) \oplus \bigoplus_{\{\sigma,\tau\} \in \binom{\Gamma}{2}_* \setminus \binom{\Gamma}{2}_{**}} (y^\sigma \cup y^\tau) \cdot (\mathbb{Z}/n\mathbb{Z})(-1) \\ &\equiv \bigoplus_{g \in \Gamma[2]^* \setminus \Gamma[2]**} \mathrm{Ind}_{\langle g \rangle}^\Gamma (y \cup y^g) \cdot (\mathbb{Z}/n\mathbb{Z})(-1), \end{aligned}$$

modulo induced modules, since  $\Gamma$  again acts freely on the first summands, while the isotropy group of an element  $\{\sigma, \tau\} \in \binom{\Gamma}{2}_* \setminus \binom{\Gamma}{2}_{**}$  is  $\langle \sigma\tau^{-1} \rangle$ . Hence

$$\begin{aligned} H^1(\Gamma, \mathrm{Br}(\overline{Q})_+^{\Gamma E}) &= \bigoplus_{g \in \Gamma[2]^* \setminus \Gamma[2]**} H^1(\langle g \rangle, \mathbb{Z}/n\mathbb{Z}(-1) \cdot (y \cup y^g)) \\ &= 0 \end{aligned}$$

by the cohomological property of  $\Gamma[2]**$  that we mentioned when introducing the notation.  $\square$

Let us now use this to prove the theorem.

*Proof of Theorem 11.* Define  $Q^{\mathrm{op}} := Q \left( \bigcup_{\tau \in \Gamma[2]**} Z_\tau \right)^{\mathrm{sing}}$  and  $X^{\mathrm{op}} := f^{-1}(Q^{\mathrm{op}})$ . Let  $j : Q^\# \rightarrow Q^{\mathrm{op}}$  and  $j' : X^\# = f^{-1}Q^\# \rightarrow X^{\mathrm{op}}$ . Consider

$$\begin{array}{ccccc} Z_\tau^{\mathrm{op}} & \xrightarrow{\iota_\tau} & Q^{\mathrm{op}} & \longleftarrow & X^{\mathrm{op}} \\ \uparrow \varphi_\tau & \nearrow & & \nwarrow \varphi'_\tau & \uparrow \\ \tilde{Z}_\tau & \longleftarrow & \tilde{Z}'_\tau & & \end{array}$$

where  $\tilde{Z}_\tau$  is a finite étale cover splitting  $y^{1/n}$  and  $\tilde{Z}'_\tau$  is the pullback. Then let

$$\begin{aligned} \mathbb{G}_{m+} &:= \left[ j_* \mathbb{G}_m \rightarrow \bigoplus_{\tau} \varphi_{\tau,*} \mathbb{Z} \right] \in \mathrm{Ch}(\mathrm{Sh}_{Q^{\mathrm{op}}}) \\ \mathbb{G}_{m+} &:= \left[ j'_* \mathbb{G}_m \rightarrow \bigoplus_{\tau} \varphi'_{\tau,*} \mathbb{Z} \right] \in \mathrm{Ch}(\mathrm{Sh}_{X^{\mathrm{op}}}) \end{aligned}$$

Now we claim that  $H^2(Q^{\mathrm{op}}, \mathbb{G}_{m+}) = \mathrm{Br}(Q)_+$ . Indeed, consider the triangle

$$\mathbb{G}_{m+} \rightarrow j_* \mathbb{G}_m \rightarrow \bigoplus_{\tau} \varphi_{\tau,*} \mathbb{Z} \xrightarrow{+1} .$$

Taking hypercohomology, we obtain

$$\bigoplus_{\tau} H^1(Q^{\text{op}}, \varphi_{\tau,*}\mathbb{Z}) \rightarrow H^2(Q^{\text{op}}, \mathbb{G}_{m+}) \rightarrow H^2(Q^{\text{op}}, j_*\mathbb{G}_m) \rightarrow \bigoplus_{\tau} H^2(Q^{\text{op}}, \varphi_{\tau,*}\mathbb{Z}).$$

Via the Leray spectral sequence the first term is zero, moreover by Hilbert 90  $R^1j_*\mathbb{G}_m = 0$  and by Gysin purity  $R^2j_*\mathbb{G}_m = 0$ , and so the second  $H^2$ -term is  $\text{Br}(Q^{\#})$ . Finally the last  $H^2$  via the Leray spectral sequence again becomes  $\bigoplus_{\tau} H^2(\tilde{Z}_{\tau}, \mathbb{Z}) = \bigoplus_{\tau} H^1(\tilde{Z}_{\tau}, \mathbb{Q}/\mathbb{Z})$ . The map  $\text{Br}(Q^{\#}) \rightarrow \bigoplus_{\tau} H^1(\tilde{Z}_{\tau}, \mathbb{Q}/\mathbb{Z})$  is the residue map, and so the identity  $H^2(Q^{\text{op}}, \mathbb{G}_{m+}) = \text{Br}(Q)_+$  follows. Analogously  $H^2(X^{\text{op}}, \mathbb{G}_{m+}) = \text{Br}(X)_+$ .

Consider

$$\mathbb{G}_{m+} \rightarrow Rf_*\mathbb{G}_{m+} \rightarrow [\mathbb{G}_{m+} \rightarrow Rf_*\mathbb{G}_m]_{+1} \xrightarrow{+1},$$

and let  $\widetilde{\tau}_{\geq 1}Rf_*\mathbb{G}_{m+} := [\mathbb{G}_{m+} \rightarrow Rf_*\mathbb{G}_m]_{+1} \xrightarrow{+1}$ . We then get

$$\text{Br}(Q)_+ \rightarrow \text{Br}(X)_+ \rightarrow H^2(Q^{\text{op}}, \widetilde{\tau}_{\geq 1}Rf_*\mathbb{G}_{m+}) \rightarrow H^3(Q^{\text{op}}, \mathbb{G}_{m+})$$

Combining with the previous we obtain

$$\begin{array}{ccccccc} \text{Br}(Q^{\text{op}}) & \longrightarrow & \text{Br}(X^{\text{op}}) & \longrightarrow & \text{Br}_h(X^{\text{op}}/Q^{\text{op}}) & \longrightarrow & H^3(Q^{\text{op}}, \mathbb{G}_m) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Br}(Q)_+ & \longrightarrow & \text{Br}(X)_+ & \longrightarrow & H^2(Q^{\text{op}}, \widetilde{\tau}_{\geq 1}Rf_*\mathbb{G}_{m+}) & \longrightarrow & H^3(Q^{\text{op}}, \mathbb{G}_{m+}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Br}(\eta) & \longrightarrow & \text{Br}(X_{\eta}) & \longrightarrow & \text{Br}_h(X_{\eta}/\eta) & \longrightarrow & H^3(\eta, \mathbb{G}_m). \end{array}$$

If we replace the second column with  $F_2\text{Br}_h$  and the last columns with the second pieces of the corresponding Hochschild–Serre filtrations, which all happen to inject into the  $(1, 2)$ -terms of the corresponding second-pages (the groups  $H^3(K, \overline{K}[Q^{\text{op}}]^{\times}) = H^3(K, \overline{K}[Q]^{\times})$ ,  $H^3(K, H^0(\overline{Q}^{\text{op}}, \mathbb{G}_{m+})) = H^3(K, \overline{K}[Q^{\text{op}}]^{\times}) = H^3(K, \overline{K}[Q]^{\times})$ ,  $H^3(K, \overline{K}[\eta]^{\times})$  vanish by Poitou–Tate, and at the same time  $\text{Pic } \overline{Q}^{\text{op}}$ ,  $H^1(Q^{\text{op}}, \mathbb{G}_{m+}) = \text{Pic } \overline{Q}^{\text{op}}$ ,  $\text{Pic } \overline{\eta}$  also all vanish), we obtain:

$$\begin{array}{ccccccc} \text{Br}(Q^{\text{op}}) & \longrightarrow & \text{Br}(X^{\text{op}}) & \longrightarrow & F_2\text{Br}_h(X^{\text{op}}/Q^{\text{op}}) & \longrightarrow & H^1(K, \text{Br}(\overline{Q})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Br}(Q)_+ & \longrightarrow & \text{Br}(X)_+ & \longrightarrow & F_2H^2(Q^{\text{op}}, \widetilde{\tau}_{\geq 1}Rf_*\mathbb{G}_{m+}) & \longrightarrow & H^1(K, \text{Br}(\overline{Q})_+) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Br}(\eta) & \longrightarrow & \text{Br}(X_{\eta}) & \longrightarrow & F_2\text{Br}_h(X_{\eta}/\eta) & \longrightarrow & H^1(K, \text{Br}(\overline{\eta})). \end{array}$$

do a diagram chase and you get the surjection claimed.  $\square$

#### MULTISECTION

Now we aim for the core of the main filtration theorem. The multisection will be a way to create local points at places of bad reduction.

**Notation 13.** For  $Q = \mathbf{R}_{E/K}\mathbb{G}_m$  with  $E/K$  Galois, we define for pairs  $(L, m)$  with  $L/E$  an étale algebra and  $\mu_m \subseteq L$ :

$$\bullet Q^{L,m} := \text{Spec} \left( L \left[ y_{\sigma}^{\pm 1/m} \right]_{\sigma \in \Gamma} \right)$$

$$\bullet Q^{L,m,(1)} := \text{Spec} \left( L \left[ y_1^{\pm 1}, y_\sigma^{\pm 1/m} \right]_{\sigma \in \Gamma^*} \right)$$

**Theorem 14.** *Let  $f : X \rightarrow Q$  be a  $T$ -equivariant RC fibration for some torus  $T$  as earlier in lecture 1. Then there exists the horizontal map in*

$$\begin{array}{ccc} Q^{\bar{K},\infty,(1)} & \xrightarrow{s} & X \\ & \searrow & \downarrow f \\ & & Q. \end{array}$$

Moreover,  $s^*(\text{Br}(X)_+) = 0$ .

**Lemma 15.** *Let  $C$  be a cyclic group acting on  $Y_{\bar{K}}$  (assumed to be smooth proper and rationally connected). Then  $Y(\bar{K})^C \neq \emptyset$ .*

*Proof.* Wlog  $C = \mu_r$ , then by Grabber–Harris–Starr there is a section of the map

$$\frac{Y \times \mathbb{G}_m}{C} \rightarrow \mathbb{G}_m/C = \mathbb{G}_m$$

This section lifts to a section of

$$Y \times \mathbb{G}_m \rightarrow \mathbb{G}_m$$

which is  $C$ -equivariant. Projecting now onto the  $Y$ -coordinate, the (limit) images of 0 and  $\infty$  are fixed points.  $\square$

*Proof of Theorem 14.* (1) We use the action of the torus. We have maps

$$\begin{array}{ccc} \bar{Q}^\infty & \longrightarrow & \bar{T} \circ \bar{X} \\ & \searrow & \downarrow \\ & & \bar{Q} \circ \bar{Q} \end{array}$$

and the right hand map is  $T$  equivariant so  $\ker(\bar{Q}^\infty \rightarrow \bar{Q}) = \widehat{\mathbb{Z}}^d$ ,  $d = [E : K]$  acts on the fibres, and contains  $\ker(\bar{Q}^\infty \rightarrow \bar{Q}^{\infty,(1)}) = \widehat{\mathbb{Z}}$ . The action factors through some  $\mathbb{Z}/m\mathbb{Z}$ , and

$$x \cdot \bar{Q}^\infty = x \cdot \overline{\bar{Q}^{\infty,(1)}}$$

provides the multisection.

(2)

$$\text{Br}(\bar{X})_+ = \text{Br}(\bar{X}) \oplus \bigoplus_{\{\sigma,\tau\} \in \binom{\Gamma}{2}_{**}} (y^\sigma \cup (y^\sigma - y^\tau)) \cdot \mathbb{Z}/n\mathbb{Z}.$$

Under the map to  $\text{Br}(\bar{Q}^{\infty,(1)})$  we lose the right hand summand.  $\square$

**Corollary 16.** *For all  $B \subseteq \text{Br}(X)_+$  finite there is a pair  $(L, m)$  with  $m$  and  $L/E$  finite and*

$$\begin{array}{ccc} Q^{L,m,(1)} & \xrightarrow{s} & X \\ & \searrow & \downarrow f \\ & & Q \end{array}$$

such that  $s^*B = 0$ .

## REDUCTION

**Proposition 17.** *Assume that the fibration theorem holds when*

- (C<sub>1</sub>)  $Q = R_{E/K} \mathbb{G}_m$  with  $E/K$  Galois and letting  $n = \max\{m : \mu_m \subseteq E\}$ .
- (C<sub>2</sub>)  $\text{gr}_2 \text{Br}_h = 0$ .
- (C<sub>3</sub>)  $\exp(\text{Br}_h) \mid n$ .
- (C<sub>4</sub>)  $\exists s : Q^{L,n,(1)} \rightarrow X$  and  $B \subseteq \text{Br}(X)_+$  with  $B \twoheadrightarrow F_1 \text{Br}_h$  such that  $s^*b = 0$ .

*Then the fibration theorem holds in general.*

## LECTURE 3

Last time we had  $f : X \rightarrow Q = \text{Res}_{E/K} \mathbb{G}_m$  with  $E/K$  finite Galois number field of degree  $d$ . We created these representatives for the image of  $\text{Br}(X_\eta)$ ,

$$\text{Br}(X)_+ \subseteq \text{Br}(X_\eta) \xrightarrow{r} \text{Br}_h(X/Q),$$

where we recall that for  $Q^\# = Q \setminus \bigcup_{\tau \in \Gamma[2]**} Z_\tau$  where  $Z_\tau$  is the Galois orbit of the geometric divisor  $\{y = y_\tau\}$ , we defined

$$\text{Br}(X)_+ := \left\{ b \in \text{Br} f^{-1}(Q^\#) : \begin{array}{l} \forall \tau \in \Gamma[2]** \\ \partial_{f^{-1}(Z_\tau)} = c_\tau \cdot (\hat{y})_{n_\tau} \\ \text{for } c_\tau \in \mathbb{Z}/n_\tau \mathbb{Z}(-1) \end{array} \right\}$$

where  $n_\tau = \max\{m : \mu_m \subseteq E^\tau\} = \max\{m : \mu_m \subseteq K(Z_\tau)\}$ . For  $L/E/K$  and  $n := \max\{r : \mu_r \subseteq E\}$  we had

$$\begin{array}{ccc} Q^{L,n,(1)} & \xrightarrow{s} & X \\ & \searrow \phi & \downarrow f \\ & & Q. \end{array}$$

where  $Q^{L,n,(1)} = \text{Spec} \left( L[y^{\pm 1}, y_\sigma^{\pm 1/n}]_{\sigma \in \Gamma^*} \right)$

## HALF-SPIN SYMBOLS

Let  $E$  be a number field containing  $\mu_n$  and  $\sigma$  be an involution on  $E$ . Take a finite set of places  $S$  of  $E$  such that  $\sigma(S) = S$  and  $S \supseteq \{w : w \mid n \infty \Delta_{E/E^\sigma}\}$ .

**Definition 18.** If  $x \in \mathcal{O}_{E,S} \setminus \{0\}$  is coprime with  $\sigma(x)$  then define

$$\text{HS}_{\sigma,S}(x) := \left( \frac{x \pmod I}{I} \right)_n$$

where  $I \subseteq \mathcal{O}_{E^\sigma,S}$  is such that  $I \mathcal{O}_{E,S} = \langle x - \sigma(x) \rangle \cdot \mathcal{O}_{E,S}$ . Note that  $x \pmod I \in (\mathcal{O}_{E,S}/I \mathcal{O}_{E,S})^\sigma = \mathcal{O}_{E^\sigma,S}/I$ , so the symbol makes sense.

**Theorem 19.** *Consider  $f : X \rightarrow Q$  and a section  $s : Q^{L,n,(1)} \rightarrow X$ . Take  $q \in \mathcal{O}_{E,S} \setminus \{0\} \subseteq Q(K)$  such that  $\tau(q)$  is coprime to  $q$  for all  $\tau \in \Gamma[2]**$ .*

*For  $b \in \text{Br}(X)_+$  such that  $s^*(b) = 0$ , and  $(P_v) \in X_q(\mathbb{A}_K)$  such that  $P_v \in S(Q^{L,n,(1)}(K_v))$  when  $q \notin Q(\mathcal{O}_v)$ , we have*

$$(b, (P_v)_v)_{\text{BM}} = \sum_{v \in S} \text{inv}_v(B(P_v)) + \sum_{\tau \in \Gamma[2]**} c_\tau \cdot \text{HS}_{\tau,s}(q).$$

*Proof.* Functoriality of residues. □

*Remark 20.* Note that  $c_\tau$  is as in the definition of  $\text{Br}(X)_+$  and in particular depends on  $b$ , and the half spin takes values in  $\mathbb{Q}/\mathbb{Z}(1)$  and  $c_\tau$  in  $\mathbb{Q}/\mathbb{Z}(-1)$ , so together we get something in  $\mathbb{Q}/\mathbb{Z}$ .

**Lemma 21** (Baby case from Smith's work). *A finite abelian group,  $a := \#A$ , and let  $M > 3a \log(a)$ . Then*

$$\exists g_0 : \{1, \dots, M-1\}^3 \rightarrow A$$

such that

$$\forall f : \{1, \dots, M\}^3 \rightarrow A$$

with  $\Delta^{(3)}f := (\partial_1 \circ \partial_2 \circ \partial_3)(f) = g_0$ , we have  $0 \in \text{im}(f)$ .

*Proof.* Assume for a contradiction that no such  $g_0$  exists. Then the triple variation map

$$\Delta^{(3)} : \text{Fun}(\{1, \dots, M\}^3 \rightarrow A \setminus \{0\}) \rightarrow \text{Fun}(\{1, \dots, M-1\}^3 \rightarrow A)$$

is surjective. The size of the domain is  $(a-1)^{M^3}$  and  $a^{M-1^3}$  is the size of the codomain. Under our assumptions on  $M, a$ , the domain is smaller than the codomain so this cannot be the case.  $\square$

**Example 22.** If  $A = \mathbb{Z}/2\mathbb{Z}$  and  $M = 2$  then  $g_0 = 1$  works.

This is the lemma we will use.  $f$  will be the Brauer manin pairing, and we'll show that we can get any triple variation from our setup. Then we'll know we must hit that  $g_0$  and some pairings must be zero.

### REDEI SYMBOLS

Let  $E$  be a number field and fix embeddings  $\bar{E} \subseteq \bar{E}_w$  for every place  $w$  of  $E$  of the corresponding algebraic closures of  $E$  and the completion. Assume that  $\mu_n \subseteq E$ .

**Definition 23.** For  $a, b, c \in E^\times$  we say that they satisfy (C) if

- (1)  $\chi_a, \chi_b, \chi_c : \Gamma_E \rightarrow \mathbb{Z}/n\mathbb{Z}(1)$  are tame including at infinity.
- (2)  $S_a, S_b, S_c$  are pairwise disjoint, where  $S_\alpha$  is the ramification locus of  $\chi_\alpha$
- (3)  $(a, b)_n = (b, c)_n = (a, c)_n = 0$  for  $(\cdot, \cdot)_n$  the order  $n$  Hilbert symbol over  $E$  (equivalently, the corresponding symbols vanish everywhere locally).

Under these assumptions, note that

$$\chi_a \cup \chi_b \cup \chi_c \in Z^3(\Gamma_E, \mathbb{Z}/n\mathbb{Z}(1))(2)$$

and since  $H^3$  of a number field is almost zero it must be a coboundary. Hence

$$\chi_a \cup \chi_b \cup \chi_c = d\xi$$

for some cochain  $\xi \in C^2(\Gamma_E, \mathbb{Z}/n\mathbb{Z}(1))(2)$ . Locally, note that

$$\chi_a \cup \chi_b \cup \chi_c|_{\Gamma_w} \in Z^3(\Gamma_w^{\text{nr}}, \mathbb{Z}/n\mathbb{Z}(1))(2)$$

and is equal then to  $d\eta_w$  for some  $\eta_w \in C^2(\Gamma_w^{\text{nr}}, \mathbb{Z}/n\mathbb{Z}(1))(2)$ .

**Definition 24.** We define the Redei symbol

$$[a, b, c]_n := \sum_w \text{inv}_w(\xi|_{\Gamma_w} - \eta_w)$$

LIFTING THROUGH  $Q^{L,n,(1)}$ 

Observe that for  $q \in E^*Q(K)$  we have  $q \in \text{im}(Q^{L,n,(1)}(K_v) \rightarrow Q(K_v))$  if and only if there is an embedding  $L \rightarrow K_v$  with  $q^\sigma \in K_v^{\times n}$  (viewing  $q^\sigma$  as an element of  $L$ . and hence  $K_v$ . through the embedding  $E \subset L$ ) for all  $\sigma \in \Gamma^*$ .

**Example 25.** When  $q \in \mathcal{O}_{E,S}$  is a principal prime and  $v$  a place of  $K$  under  $q$ , then the second condition above means there is a place  $w$  associated to  $q$  which is completely split over  $K$  and  $w$  splits in  $L$  and

$$\left( \frac{\sigma(q)}{q} \right)_{v,n} = 1$$

for all  $\sigma \in \Gamma^*$ , where this is the spin symbol.

**Definition 26.**  $q \in Q(K)$  is  $(L, n)$  liftable outside of  $S$  if and only if  $q \in Q(\mathcal{O}_v) \cup \text{im}(Q^{L,n,(1)}(K_v))$  for all  $v \notin S$ .

**Definition 27** (Grids). A grid of dimension  $r$  and size  $M$  is an injective function

$$\mathcal{M} : \{1, \dots, M\}^r \rightarrow \mathcal{O}_{E,S} \setminus \{0\} \subseteq Q(K),$$

of the shape

$$\mathcal{M}_{i_1, \dots, i_r} = a_0 \cdot a_{i_1,1} \cdots a_{i_r,r}$$

where  $a_0 \in \mathcal{O}_{E,S}$  and  $R_i := \{a_{1,i}, \dots, a_{M,i}\} \subseteq \mathcal{O}_{E,S}$  are ordered subsets (think of  $R_i$  as the row of the grid).

We identify  $\mathcal{M} = a_0 \cdot \{a_{1,1}, \dots, a_{M,1}\} \cdots \{a_{r,1}, \dots, a_{M,r}\}$ .

Say  $\mathcal{M}$  is

- squarefree over  $K$  if  $N_{E/K}(x)$  is squarefree for all  $x \in \mathcal{M}$ .
- prime-var if all  $a_{i,j}$  are prime elements such that  $N_{E/K}(a_{i',j'})$  and  $N_{E/K}(a_{j,i})$  are coprime in  $\mathcal{O}_{K,S}$  for all  $(i, j) \neq (i', j')$ .

**Definition 28.** A grid  $\mathcal{M}$  as above is  $(L, n)$ -liftable outside  $S$  if all elements of  $\mathcal{M}$  are and for all  $v \notin S$  and all  $q, q' \in \mathcal{M} \setminus Q(\mathcal{O}_v)$  we have  $q'q^{-1} \in \text{im}(Q^{L,n}(K_v))$ .

We define notations

- $B_L := \ker(\text{Br}_e Q \rightarrow \text{Br} Q^{L,n,(1)})$
- $B'_L := B_L + \langle \text{cores}_{E/K}(y \cup (y_\sigma - y))_{n_\sigma} \rangle_{\sigma \in \Gamma[2]**}$ .

**Theorem 29.**  $(P_v)_{v \in S} \in Q(K_\Omega)^{B'_L}$ . For all  $M \in \mathbb{N}$  and  $g : \{1, \dots, M-1\}^3 \times \Gamma^2 \rightarrow \mathbb{Z}/n\mathbb{Z}(2)$  there exists a prime-var  $(L, n)$ -liftable grid

$$\mathcal{M} = q_0 \cdot \{q_{1,1}, \dots, q_{M,1}\} \cdots \{q_{1,3}, \dots, q_{M,3}\}$$

such that

- all  $q \in \mathcal{M}$  approximate  $(P_v)$  for  $v \in S$
- the “direction” in  $(E \otimes \mathbb{R})^\times / \mathbb{R}_{>0} = \prod_{v|\infty} Q(K_v) / \mathbb{R}_{>0}$  of all  $q \in \mathcal{M}$  approximate that of  $(P_v)_{v|\infty}$
- $\text{HS}_{S,\tau}(q) = 1$  for all  $\tau \in \Gamma[2]**, q \in \mathcal{M}$ .
- $[q_{i+1,1}q_{i,1}^{-1}, \sigma(q_{j+1,1}q_{j,1}^{-1}), \tau(q_{k+1,1}q_{k,1}^{-1})] = g(i, j, h)(\sigma, \tau)$

*Proof.* There is not time for the proof, but it builds on work of Shafarevich.  $\square$

## ADELIC LIFTS

Take  $\mathcal{M}$  an  $(L, n)$ -liftable outside of a large set  $S$ . Let  $U_v \ni P_v$  be an open neighbourhood. Then take the notation that for  $q \in \mathcal{M}$  we will write

$$X_q(K_v) \ni \mathcal{P}_v(q) := \begin{cases} \in U_v & \forall v \in S \\ \in \mathcal{X}_q(\mathcal{O}_v) & \forall v \notin S \text{ such that } q \in Q(\mathcal{O}_v) \\ \in s(q^{L,n}) & \forall v \notin S \text{ such that } q \notin Q(\mathcal{O}_v) \end{cases}$$

and insist that  $q^{L,n}$  is in some way multiplicative in  $q$ . By  $q^{L,n}$  we mean an element such that  $p(q^{L,n}) = q$  in the diagram

$$\begin{array}{ccc} Q^{L,n,(1)} & \xrightarrow{s} & X \\ & \searrow p & \downarrow f \\ & & Q \end{array}$$

## VARIATION THEOREM

**Notation 30.** Let  $\bar{\beta} \in H^3(\bar{Q}, \mathbb{G}_m)[n]^{\Gamma\kappa} = H^3(\bar{Q}, \mu_n)^{\Gamma\kappa}$ . Take a total ordering  $<$  on  $\Gamma$  and write

$$\bar{\beta} = \sum_{\sigma < \tau < \rho} a_{\sigma,\tau,\rho} \cdot (y_\sigma \cup y_\tau \cup y_\rho)_n$$

where the coefficients  $a_{\sigma,\tau,\rho} \in \mathbb{Z}/n\mathbb{Z}(-2)$ .

- Extend beyond the ordered triple setting by making it alternating in  $(\sigma, \tau, \rho)$ .
- $a_{\sigma g, \tau g, \rho g} = a_{\sigma, \tau, \rho}$ .

$$R_{\bar{\beta}}[x_1, x_2, x_3]_n := \sum_{(\sigma, \tau, \rho) \in \Gamma^3/\Gamma} a_{\sigma, \tau, \rho} [x_1^\sigma, x_2^\tau, x_3^\rho]_n \in \mathbb{Z}/n\mathbb{Z}$$

**Theorem 31.** Let  $b \in \text{Br}_h$ . After possibly replacing the field  $L$  defining  $s$  with an extension, the following hold.

For all  $(P_v)_{v \in S}$ ,

- $\forall U_v \ni P_v$  arbitrarily small neighbourhood for  $v \in S$  and  $v \nmid \infty$ .
- $\forall U_v \ni P_v$  open connected such that  $f(U_v)$  is  $\mathbb{R}_{>0}$ -inv arbitrarily small for  $v \mid \infty$ .

For all  $\mathcal{M} = q_0 \cdot \{q_{1,1}, q_{2,1}\} \cdot \{q_{1,2}, q_{2,2}\} \cdot \{q_{1,3}, q_{2,3}\}$  prime-var and  $(L, n)$ -liftable.

$$\sum_{q \in \mathcal{M}}^{\text{altern}} (b, (\mathcal{P}_v(q)))_{\text{BM}} = R_{\bar{\partial}(b)}[q_{2,1}q_{1,1}^{-1}, q_{2,2}q_{1,2}^{-1}, q_{2,3}q_{1,3}^{-1}]_n.$$

where here  $\bar{\partial}(b)$  is the image of  $b \in \text{Br}_h$  after mapping through the composition

$$\text{Br}_h \rightarrow H^3(Q, \mathbb{G}_m) \rightarrow H^3(\bar{Q}, \mathbb{G}_m)^{\Gamma\kappa}$$

*Proof.* (Very) long computation with Čech cocycles.  $\square$

Now we prove the fibration theorem! Take the dual of the third graded piece.

$$A = (F_3 \text{Br}_h / F_2 \text{Br}_h)^D = \text{coker}(\text{Br}(X_\eta \rightarrow \text{Br}_h))^D.$$

Apply Smith's lemma and get the existence of a function

$$g_0 : \{1, \dots, M-1\}^3 \rightarrow (F_3/F_2)^D$$

such that for all  $f$  with  $\Delta^{(3)}f = g_0$ ,  $0 \in \text{im}(f)$ .

Consider the composition

$$F_3\text{Br}_h/F_2\text{Br}_h \xrightarrow{\bar{\delta}} H^3(\bar{Q}, \mathbb{G}_m)[n]^{\Gamma_K} \rightarrow \text{Fun}(\Gamma^2, \mathbb{Z}/n\mathbb{Z}(2))$$

where the latter map is  $\sum a_{\sigma,\tau,\rho} \cdot (y_\sigma \cup y_\tau \cup y_\rho) \mapsto [(\sigma, \tau) \mapsto a_{1,\sigma,\tau}]$ . This composition is an injection. Dualise it to get a surjection and then lift  $g_0$  to a function

$$g : \{1, \dots, M-1\}^3 \rightarrow \text{Fun}(\Gamma^2, \mathbb{Z}/n\mathbb{Z}(2)).$$

In Theorem 29 put this  $g$ . By Theorem 31, the function

$$f : \{1, \dots, M\}^3 \rightarrow (F_3/F_2)^D, (i, j, k) \mapsto (-, (\mathcal{P}_v(q(i, j, k)))_v)_{\text{BM}}$$

with  $q(i, j, k) := q_0 q_{i,1} q_{j,2} q_{k,3}$ , has triple variation  $g_0$ .

## APPENDIX A. SOME LEMMATA

Here are some things we worked out along the way.

**Lemma 32.** *Let  $X$  be a finitely generated Galois module. Then there exists a short exact sequence of Galois modules*

$$0 \rightarrow M \rightarrow N \rightarrow X \rightarrow 0$$

where  $M, N$  are free as abelian groups and  $N$  is a permutation module.

*Proof.* Assume that the Galois action on  $X$  factors through the finite quotient  $\Gamma$ . Then let  $G$  be a generating set for  $X$  and  $N := \bigoplus_{x \in G} \mathbb{Z}[\Gamma]$ . This surjects via the obvious map onto  $X$  as Galois modules and is itself a permutation module which is free as an abelian group.  $\square$

**Lemma 33.** *Let  $G$  be a finite abelian group identified with the associated finite group scheme over  $K$ . Then there is a short exact sequence in the Zariski topology*

$$0 \rightarrow G \rightarrow T \rightarrow Q \rightarrow 0$$

where  $T/K$  is a torus and  $Q$  is a quasitrivial torus

*Proof.* Let  $X = \text{Hom}(G, \mathbb{G}_m)$  be the associated character module. Then via the general theory of tori and their characters we have to exhibit a short exact sequence of Galois modules

$$0 \rightarrow A \rightarrow B \rightarrow X \rightarrow 0$$

where  $A, B$  are free finite rank abelian groups and, since we want  $Q$  to be quasitrivial,  $A$  is a permutation module. By Lemma 32 we have a short exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow X \rightarrow 0$$

where  $M, N$  are free as abelian groups and  $N$  is a permutation module. Similarly we have such a sequence for  $\text{Hom}(M, \mathbb{Z})$

$$0 \rightarrow M' \rightarrow N' \rightarrow \text{Hom}(M, \mathbb{Z}) \rightarrow 0.$$

applying  $\text{Hom}(-, \mathbb{Z})$  to this latter sequence we get an exact sequence

$$0 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 0$$

where now  $P = \text{Hom}(N', \mathbb{Z})$  is a torsionfree permutation module and  $Q = \text{Hom}(M', \mathbb{Z})$  is a torsionfree module. Let  $B := P \oplus N / \langle (m, -m) : m \in M \rangle$ . Then since  $B/N \cong P/M \cong Q$  and both  $N, Q$  are torsionfree,  $B$  must be torsionfree. Taking  $A := P$  which is a torsionfree permutation module, we then have an exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow X \rightarrow 0$$

as required.  $\square$