CHARACTERISTIC P PHENOMENA IN BRAUER GROUPS

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Disclaimer. These notes were taken live during lectures. In particular, any mistakes are the fault of the transcriber and not of the lecturer.

Lecture 1 (Bright): Title

1. Setting the scene

Today, everything is "standard material", meaning at least 30 years old, but is to gather things in place and set the scene for what will later come.

- Let X be a geometrically irreducible variety over a field k which is of characteristic 0 or is perfect of characteristic p.
- If k is a discretely valued field of characteristic 0, we write $\mathcal{O}_k$ for the ring of integers, $\mathbb{F}$ for the residue field of characteristic p, $\mathcal{X}/\mathcal{O}_K$ for a scheme with generic fibre X/K and special fibre $Y/\mathbb{F}$.

Our aim will be to study the Brauer group $\mathrm{Br}(X) := H^2(X, \mathbb{G}_m) \subseteq \mathrm{Br}(k(X))$ where here this cohomology is étale or equivalently flat.

1.1. **Recall the situation in characteristic 0.** Recall the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0,$$

This relates $\mathrm{Br}(X)$ to étale cohomology groups with constructible coefficients (i.e. μ_n, which is a particularly nice one!). In particular it gives the short exact sequence

$$0 \to \mathrm{Pic}(X)/n\mathrm{Pic}(X) \to H^2(X, \mu_n) \to \mathrm{Br}(X)[n] \to 0.$$

Below are some fundamental theorems in this context.

- *finiteness*: $H^2(\overline{X}, \mu_n) \cong H^2(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ is finite.
- *Proper base change*: $H^2(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong H^2(X_K, \mathbb{Z}/n\mathbb{Z})$ for $\overline{k} \subseteq K$ algebraically closed.
- *Comparison*:

  $$H^2(X_{\mathbb{C}}, \mathbb{Z}/n\mathbb{Z}) \cong H^2_{\mathrm{an}}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}).$$

  Combining these we can prove

  $$0 \to (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \to \mathrm{Br}(\overline{X}) \to H^3_{\mathrm{an}}(X(\mathbb{C}), \mathbb{Z})_{\mathrm{tors}} \to 0$$
- *smooth base change*: $H^2(\overline{X}, \mu_n)$ is constant on geometric fibres of a smooth family.
- *Gysin sequence*: if $D \subseteq X$ is a smooth prime divisor then we have an exact sequence

  $$0 \to \mathrm{Br}(X) \to \mathrm{Br}(X \backslash D) \xrightarrow{\partial_D} H^1(D, \mathbb{Q}/\mathbb{Z}).$$

Now for characteristic p! When $\mathrm{char}(k) = p$ and $p \nmid n$, these statements above all still hold and describe $\mathrm{Br}(X)(p')$ (the prime to p part).

Also in mixed characteristic, e.g. smooth base change for $\mathcal{X}/\mathcal{O}_K$ smooth and proper gives

$$H^2(\overline{Y}, \mu_n) \cong H^2(\overline{X}, \mu_n),$$

and we get a Gysin sequence

$$0 \to \mathrm{Br}(\mathcal{X})(p') \to \mathrm{Br}(X)(p') \to H^1(Y, \mathbb{Q}/\mathbb{Z}).$$

This is useful in studying the evaluation map

$$X(K) \xrightarrow{\mathcal{A}} \mathrm{Br}(k)$$

for $\mathcal{A} \in \mathrm{Br}(X)$ by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(\mathcal{X})(p') \arrow[r] & \mathrm{Br}(X)(p') \arrow[r] \arrow[d, "\mathrm{ev}_P"] & H^1(Y, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\mathrm{ev}_{P_0}"] \\
 & & \mathrm{Br}(k) \arrow[r, equal] & H^1(F, \mathbb{Q}/\mathbb{Z})
\end{tikzcd}$$

for $P \in X(k)$ reducing to $P_0 \in Y(\mathbb{F})$

1.2. **What goes wrong for the p-part?**

Let X be smooth and geometrically integral over k with characteristic p. The Kummer sequence fails now to be exact in the étale topology, and $H^2(X_{\mathrm{et}}, \mu_p)$ is not useful. We can use $H^2(X_{\mathrm{fppf}}, \mu_p)$ instead. The Artin–Schreier sequence

$$0 \to \mathbb{Z}/p \to \mathcal{O}_X \xrightarrow{1-F} \mathcal{O}_X \to 0$$

then gives us ways to compute. Moreover, $H^i(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > \dim(X) + 1$, and $H^i(X_{\mathrm{fppf}}, \mu_p)$ contains the k-points of a unipotent group.

If we're in the mixed characteristic setting, things still go wrong! Indeed, for $\mathcal{X}/\mathcal{O}_k$ with $\mathrm{char}(\mathbb{F}) = p$ the Gysin sequence fails because there's no way to define ∂_Y on $\mathrm{Br}(X)[p]$. Moreover, smooth base change fails e.g. by [BMS18] where the authors give two smooth proper 3-folds over $\mathbb{Q}_2$ with the same smooth reduction but different $H^2(-, \mathbb{Z}/2)$.

One conclusion you could draw from this is that characteristic p is hopeless!

1.3. **Differentials and the Cartier operator.**

Let X be smooth over a perfect field k of characteristic p. We have the de Rham complex

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \to \dots$$

where we define $Z^i := \ker(d)$ and $B^i = \mathrm{im}(d)$, each in the ith position. Note that d is *not* a map of $\mathcal{O}_X$-modules. But something odd happens in characteristic p:

$$d(f^p\omega) = pf^{p-1}d(f) \wedge \omega + f^p d(\omega) = f^p d(\omega).$$

So it is linear if we let $\mathcal{O}_X$ act through $f \mapsto f^p$. Indeed, then Z^i and B^i are then $\mathcal{O}_X$-submodules.

Definition 1. Define $\mathrm{dlog} : \mathcal{O}_X^\times \to \Omega^1_{X/K}$ by $\mathrm{dlog}(f) = \frac{df}{f}$.

This map is a group homomorphism via the Leibniz rule:

$$\mathrm{dlog}(fg) = \frac{d(fg)}{fg} = \frac{fd(g) + gd(f)}{fg} = \mathrm{dlog}(f) + \mathrm{dlog}(g)$$

Moreover in the étale topology we define $\Omega^1_{X/k,\mathrm{dlog}} := \mathrm{im}(\mathrm{dlog})$ and can show that $\ker(\mathrm{dlog}) = (\mathcal{O}_X^\times)^p$. Hence we have a short exact sequence

$$0 \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to \Omega^1_{X/k,\mathrm{dlog}} \to 0$$

We will use this to replace the Kummer sequence.

Cartier defined a map $C : Z^i/B^i \to \Omega^i_{X/k}$ (yes, a subquotient mapping back to the whole!), satisfying

$$C\left(x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_i}{y_i}\right) = x\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_i}{y_i}.$$

This map C is an isomorphism. The map going in the other direction is a bit like a Frobenius operator, morally.

Theorem 2 (Cartier). *$\omega \in \Omega^1_{X/k}$ is logarithmic if and only if $d\omega = 0$ and $C(\omega) = \omega$.*

An excellent survey of this is Colliot-Thélène's survey on the work of Bloch–Kato. We have a commutative diagram

$$\begin{tikzcd}
Z^i \arrow[r, "C-1"] \arrow[d, "C"] & \Omega^i_{X/K} \arrow[d] \\
\Omega^i_{X/k} \arrow[r, "1-C^{-1}"] & \Omega^i_{X/k}/B^i.
\end{tikzcd}$$

which gives

$$0 \to \Omega^1_{X/k,\mathrm{dlog}} \to \Omega^1_{X/k} \xrightarrow{1-C^{-1}} \Omega^1_{X/k}/B^1 \to 0.$$

Milne defined $\nu(i) := \ker\left(Z^i \xrightarrow{C-1} \Omega^i_{X/K}\right)$, note that $\nu(1) = \Omega^1_{X/k,\mathrm{dlog}}$.

Bloch showed that $\nu(i)$ is generated locally by elements of the form $\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_i}{y_i}$.

Let $\Omega^i_{\log} := \nu(i)$ (note that $\nu(0) = \mathbb{Z}/p$), which will be our higher degree analogue of logarithmic differentials. As above we get

$$0 \to \Omega^i_{\log} \to \Omega^i \xrightarrow{1-c^{-1}} \Omega^i/B^i \to 0$$

For $i = 0$ this is the Artin–Schreier sequence! Now, for a notation we define

$$\mathbb{Z}/p\mathbb{Z}(r) := \Omega^r_{\log}[-r],$$

where by the $[-r]$ we just mean degree shifting in the appropriate derived category. Thus, entirely formally, we have

- $H^r(Z, \mathbb{Z}/p\mathbb{Z}(r)) = H^0(X, \Omega^r_{\log})$
- $H^r(Z, \mathbb{Z}/p\mathbb{Z}(r-1)) = H^1(X, \Omega^r_{\log})$

1.4. **The de Rham Witt complex.** Now we want to see what happens for higher powers of p. It's complicated, but we'll see only what we need! Recall the following.

Definition 3. Let A be a ring of characteristic p, then we will define rings $W_n(A)$ and $W(A) = \lim\limits_{\leftarrow} W_n$. The satisfy

- $W_1(A) := A$
- Teichmüller map: $x \in A \mapsto \overline{x} \in W(A)$
- There are maps
  - $V : W(A) \to W(A)$
  - $F : W(A) \to W(A)$ lifting Frobenius
  - $VF = FV = p$.

These were introduced by Witt (1937) in order to generalise Artin–Schreier theory.

These sheafify nicely so we get $W_n\mathcal{O}_X$ and $W\mathcal{O}_X$ and an exact sequence in the étale topology

$$0 \to \mathbb{Z}/p^n \to W_n\mathcal{O}_X \xrightarrow{1-F} W_n\mathcal{O}_X \to 0$$

Theorem 4 (Deligne–Illusie 1979)**.** *Define the de Rham–Witt complex: a projective system of complexes $W_n\Omega^i$ (one for each n) with differentials induced by d in some way and the following properties*

- $W_n\Omega^0_X = W_n\mathcal{O}_X$.
- $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$.
- $F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$.
- $FV = VF = p$
- $Fd\underline{x} = \underline{x}^{p-1}d\underline{x}$, *i.e. F extends inverse Cartier.*

These complexes all package together as $W\Omega^i_X := \lim\limits_{\leftarrow} W_n\Omega^i_X$.

Deligne–Illusie's motivation for this is the slope spectral sequence

$$E_1^{ij} = H^j(X, W\Omega^i_X) \implies H^{i+j}(X/W)$$

where the right hand side is crystalline cohomology, and $W = W(k)$.

Definition 5. We define

$$\operatorname{dlog} : \mathcal{O}_X^\times \to W_n\Omega^1_X$$

by $\operatorname{dlog}(f) = \frac{d\underline{f}}{\underline{f}}$. We define $W_n\Omega^1_{\log}$ for the image of dlog. This gives us a short exact sequence

$$0 \to \mathcal{O}_X^\times \xrightarrow{p^n} \mathcal{O}_X^\times \xrightarrow{\operatorname{dlog}} W_n\Omega^1_{\log} \to 0.$$

We then take the notation $\mathbb{Z}/n\mathbb{Z}(r)$ by writing $n = p^s m$ for m coprime to p and then

$$\mathbb{Z}/n\mathbb{Z}(r) := \mu_m^{\otimes r} \oplus W_s\Omega^r_{X,\log}[-r].$$

Then

$$0 \to \mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \xrightarrow{[1]}$$

is a triangle in the étale topology for all n (X either characteristic 0 or smooth over a perfect field of characteristic p).

1.5. **Okay, but what about Artin–Schreier?** We have

$$0 \to W_\bullet \Omega^i_{X,\log} \to W_\bullet \Omega^i_X \xrightarrow{1-F} W_\bullet \Omega^i_X \to 0.$$

For fixed n

$$0 \to W_n \Omega^i_{X,\log} \to W_n \Omega^i_X \xrightarrow{1-F} W_n \Omega^i_X / dV^{n-1} \Omega^{i-1}_X \to 0.$$

In particular, we get

$$H^0(X, W_n \Omega^1_X) \twoheadrightarrow H^1(X, W_n \Omega^1_{\log}) = H^2(X, \mathbb{Z}/p^n(1)) \twoheadrightarrow \mathrm{Br}(X)[p^n].$$

For $n = 1$ we have $x\frac{dy}{y} \mapsto [x, y)$ (in the notation of Serre's local fields) which is a cup product.

1.6. **Relation to flat cohomology.** We left flat cohomology hanging earlier, what's going on there? Well we need to understand $H^2_{\mathrm{fppf}}(X, \mu_{p^n})$. Let us consider the usual morphism of sites $\varepsilon : X_{\mathrm{fppf}} \to X_{\mathrm{et}}$, then we get a Leray spectral sequence

$$E_2^{ij} : H^i(X_{\mathrm{et}}, R^j \varepsilon_* \mu_{p^n}) \implies H^{i+j}(X_{\mathrm{fppf}}, \mu_{p^n}).$$

Use the Kummer sequence

$$0 \to \mu_{p^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

on X_{fppf}. We have $\varepsilon_* \mathbb{G}_m = \mathbb{G}_m$ and $R^j \varepsilon_* \mathbb{G}_m = 0$ for $j > 0$. So

$$0 \to \varepsilon_* \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to R^1 \varepsilon_* \mu_{p^n} \to 0$$

on X_{et}. Hence $\varepsilon_* \mu_{p^n} = 1$ and $R^1 \varepsilon_* \mu_{p^n} \cong W_n \Omega^1_{X,\log}$ and hence

$$H^i(X_{\mathrm{fppf}}, \mu_{p^n}) \cong H^{i-1}(X, W_n \Omega^1_{X,\log}).$$

1.7. **Milnor K-theory and the Bloch–Kato Conjecture.** Let K be any field, and define the Milnor K-theory of K to be

$$K_0(K) = \mathbb{Z}$$
$$K_1(K) = K^\times$$
$$K_r(K) = (K^\times)^{\otimes r} / \langle x_1 \otimes \cdots \otimes x_r \ : \ x_i + x_j = 1 \ \forall i \neq j \rangle$$

As a matter of notation, we will write $\{x_1, \ldots, x_r\} \in K_r(K)$ for the class of the elementary tensor $x_1 \otimes \cdots \otimes x_r$. If $\mathrm{char}(K) \nmid n$ then

$$K^\times / K^{\times n} \xrightarrow{h_n} H^1(K, \mu_n),$$

and moreover we can extend via the cup product to

$$K_r(K)/nK_r(K) \xrightarrow{h_n^r} H^r(K, \mu_n^{\otimes r}).$$

- Bloch–Kato conjecture: h_n^r is an isomorphism
- Proved by Voevodsky, Rost, Suslin, ... and in the case $r = 2$ this was shown by Merkurjev–Suslin.

If $\mathrm{char}(K) = p$ then we define a different map

$$\psi^r : K_r(K) \to \Omega^r_K$$
$$\{x_1, \ldots, x_r\} \mapsto \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{x_r}$$

The image is contained in $\nu(r) := \ker \left(\Omega^r \xrightarrow{1-c^{-1}} \Omega^r_K / B^r \right)$.

Theorem 6 (Bloch–Gabber–Kato)**.** *$K_r(K)/p^n \to W_n\Omega^r_{K,\log} \to H^r(K,\mathbb{Z}/p^n\mathbb{Z}(r))$ is an isomorphism*

LECTURE 2 (MOLYAKOV)

2. RAMIFICATION THEORY OF HENSELIAN DVRS

2.1. **Classical case.** Let K be a local field and L/K be finite Galois with $\mathrm{Gal}(L/K) =: G$. Then we recall the lower numbered ramification filtration

$$G \geq G_0 \geq G_1 \geq \dots$$

where

$$G_i := \ker\left(G \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^{i+1})\right).$$

There is then the upper numbering of these groups, constructed using the Herbrand function ψ which is defined as the inverse to

$$\begin{aligned}\psi^{-1} : [-1,\infty) &\to [-1,\infty)\\ u &\mapsto \int_0^u \frac{dt}{[G_0 : G_t]}.\end{aligned}$$

The upper numbering is then defined by $G^i := G_{\psi(i)}$. The upper numbering is well behaved in towers in ways that the lower numbering is not. In particular we can construct from limiting the upper numbering:

- $\left\{\Gamma_K^i\right\}$ a filtration on $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$.
- a filtration on $H^1(K,\mathbb{Q}/\mathbb{Z})$.

Theorem 7. *(Hasse–Arf) $\mathrm{fil}_i H^1(K,\mathbb{Q}/\mathbb{Z}) = \mathrm{fil}_{\lceil i\rceil} H^1(K,\mathbb{Q}/\mathbb{Z})$.*

There is also local Tate duality:

Theorem 8. *There is a perfect pairing*

$$H^1(K,\mathbb{Z}/n\mathbb{Z}) \times H^1(K,\mathbb{Z}/n\mathbb{Z}(1)) \to \mathrm{Br}(K)[n] \cong \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Using this pairing, the ramification filtration on $H^1(K,\mathbb{Z}/n\mathbb{Z})$ is the annihilator of the image of $U^{(i+1)}$ under the Kummer map $K^\times \to H^1(K,\mathbb{Z}/n\mathbb{Z}(1))$. That is,

$$\mathrm{fil}_i H^1(K,\mathbb{Z}/n\mathbb{Z}) := \mathrm{Ann}(\mathrm{im}(U^{(i+1)})).$$

Now for some remarks.

Remark 9. The following should be observed.

- The classical theory works for K a complete discretely valued field with perfect residue field.
- Let R be a Henselian DVR, $\widehat{R}$ its completion, $K = \mathrm{Frac}(R)$, $\widehat{K} := \mathrm{Frac}(\widehat{R})$. There is a natural bijection

  $$\left\{\begin{smallmatrix}\text{finite separable}\\ \text{extensions of } K\end{smallmatrix}\right\} \leftrightarrow \left\{\begin{smallmatrix}\text{finite separable}\\ \text{extensions of } \widehat{K}\end{smallmatrix}\right\}$$

  preserving Galois exactly, i.e. $\Gamma_K = \Gamma_{\widehat{K}}$.

Geometrically we will be interested in the behaviour of schemes over local rings and their special fibres, which fit into a diagram of the shape below.

$$\begin{tikzcd}
\operatorname{Spec}(K_v) \arrow[r] \arrow[d] & \operatorname{Spec}(\mathcal{O}_{\mathcal{X},\nu}) \arrow[d] & \nu \arrow[l] \arrow[d] \\
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & Y \arrow[l] \arrow[d] \\
\operatorname{Spec}(\mathbb{Q}_p) \arrow[r] & \operatorname{Spec}(\mathbb{Z}_p) & \mathbb{F}_p \arrow[l]
\end{tikzcd}$$

2.2. **Kato's Swan conductor.** Let R be a Henselian DVR and $K := \operatorname{Frac}(R)$ with $R/\mathfrak{m} = F$ the residue field. We will write

$$\begin{aligned} H^q_n(K) &= H^q(K, \mathbb{Z}/n\mathbb{Z}(q-1)) \\ H^q(K) &= \lim_{\to_n} H^q_n(K) \end{aligned}$$

Fact: It is a fact, but a very difficult one, that $H^q_n(K) \to H^q(K)$ is injective.

Moreover, if we take the Kummer map $K^\times \to H^1(K, \mathbb{Z}/n\mathbb{Z}(1))$ then we can use the cup product map to construct pairings

$$\begin{aligned} \{\cdot,\cdot\} &: H^q_n(K) \times K^\times \to H^{q+1}_n(K) \\ \{\cdot,\cdot\} &: H^q(K) \times K^\times \to H^{q+1}(K). \end{aligned}$$

Definition 10. We define a filtration

$$\operatorname{fil}_i H^q(K) := \left\{ \chi \in H^q(K) \ : \ \begin{smallmatrix} \forall \text{ Henselian DV fields} \\ K'/K \text{ s.t. } R' \supset R, \mathfrak{m}' \supset \mathfrak{m} \\ \text{we have } \left\{\chi, 1+\pi_{K'} R'\right\} = 0 \\ \text{in } H^{q+1}(K') \end{smallmatrix} \right\}$$

Moreover, the Swan conductor of $\chi \in H^1(K)$ is defined to be

$$\operatorname{sw}(\chi) := \min\{i \ : \ \chi \in \operatorname{fil}_i H^q(K)\}$$

Remark 11. Let $\tilde{R} := R[T]^h_{(\pi)}$ and $\widehat{K} := \operatorname{Frac}(\tilde{R})$ then $\chi \in \operatorname{fil}_i H^q(K)$ if and only if $\left\{\chi, 1 + \pi^{i+1} T\right\} = 0$ in $H^{q+1}(K)$.

Proposition 12. *The following are true.*

1. *The prime-to-characteristic-of-K part of* $H^q(K)$ *is contained in* $\operatorname{fil}_0 H^q(K)$.
2. $H^q(K) = \bigcup_i \operatorname{fil}_i H^q(K)$

Proof.

1. $U^{(1)} \subseteq R^{\times r}$ for all r coprime to the residue characteristic by Hensel.
2. If $\operatorname{char}(K) = 0$ and $\operatorname{char}(F) = p$, then we need to check that the p-primary part lives in a piece of the filtration. Let $e = \operatorname{ord}_\pi(p)$.

   Lemma 13. *For any* $i > \frac{e}{p-1}$ *we have* $U^{(i+e)} = (U^{(i)})^p$.

   Corollary 14. $\operatorname{fil}_i H^q_{p^n}(K) = H^q_{p^n}(K)$ *for* $i > \frac{e}{p-1}$.

   Hence the result follows.

□

Theorem 15 (Kato)**.** *There is an exact sequence*

$$0 \to H^q(K) \xrightarrow{\iota_q} \operatorname{fil}_0 H^q(F) \xrightarrow{\mathrm{res}} H^{q-1}(F) \to 0$$

where:

- ι_q *is defined by*
$$\iota_q : H^1(F, \mathbb{Q}/\mathbb{Z}) \to H^1(K, \mathbb{Q}/\mathbb{Z})$$
$$\iota_q \left\{\chi, \overline{a}\right\} = \left\{\iota^{q-1}(\chi), a\right\}$$
for all $\chi \in H^{q-1}(F)$ *and* $a \in R^\times$*.*
- res *is defined by*
$$\mathrm{res} \left\{\iota^{q-1}(\chi), \pi\right\} = \chi$$
for all $\chi \in H^{q-1}(F)$*.*

Example 16. [R:I missed an example involving $\Omega_{\log}$ and Ω^q_F/B^q]

It is almost a corollary of the theorem that the zeroth term in the filtration then sees ramification in the following sense.

Corollary 17. $\operatorname{fil}_0 H^q(K) \supseteq \ker(H^q(K) \to H^q(K^{\mathrm{nr}}))$

Example 18. $q = 2$, then $H^2(K) = \mathrm{Br}(K)$ and $\operatorname{fil}_0 H^2(K) = \mathrm{Br}(K^{\mathrm{nr}}/K)$. We get

$$\begin{tikzcd}
0 \arrow[r] & R^{\mathrm{nr},\times} \arrow[r] & K^{\mathrm{nr},\times} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & H^2(F, R^{\mathrm{nr},\times}) \arrow[r] \arrow[d, equal] & \operatorname{fil}_0 H^2(K) \arrow[r] & H^2(F, \mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
 & H^2(F) & & H^1(F, \mathbb{Q}/\mathbb{Z}) &
\end{tikzcd}$$

Let us now define Kato's swan conductor.

$$\Omega^q_R(\log) := \Omega^q_R + \Omega^{q-1}_R \wedge \frac{d\pi}{\pi} \subseteq \Omega^q_K$$
$$\omega^q_F := \Omega^q_R(\log) \otimes_R F$$

Then we have noncanonically split exact sequences with vertical maps all being surjective below.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^q(F) \arrow[r] \arrow[d] & \omega^q_F \arrow[r] \arrow[d] & \Omega^{q-1}(F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{q-1}_p(F) \arrow[r] & \operatorname{fil}_0 H^{q-1}_p(K) \arrow[r] & H^q_p(F) \arrow[r] & 0
\end{tikzcd}$$

Theorem 19. *Let* $\chi \in \operatorname{fil}_i H^q(K)$ *for* $i \geq 1$*, then there exists a unique* $\eta \in \mathfrak{m}_R^{-i} \otimes \omega^q_F$ *such that for all Henselian DV extensions* R'/R *we have* $\{\chi, 1+z\} = \lambda_{R'}(z\eta)$ *for all* $z \in \mathfrak{m}^i_R R'$*. Moreover,* $\chi \mapsto \eta$ *induces an injection*

$$\mathrm{gr}_i : H^q(K) \to \mathfrak{m}_R^{-i} \otimes \omega^q_F$$

LECTURE 3 (BRIGHT): EVALUATING THE WILD BRAUER GROUP

This is concerning joint work with Rachel Newton. Let K be p-adic field with integers $\mathcal{O}_K$, residue field $\mathbb{F}$, uniformiser π. Let $\mathcal{X}/\mathcal{O}_K$ be smooth with geometrically integral fibres and write X/K for the generic fibre and $Y/\mathbb{F}$ for the special fibre.

Definition 20. Now we will define the *evaluation filtration* on $\mathrm{Br}X$ as follows. For $n \geq 0$ we define

$$\mathrm{Ev}_n \,\mathrm{Br}X = \left\{ A \in \mathrm{Br}X \;:\; \begin{smallmatrix} \forall K'/K \text{ finite } \forall P \in \mathcal{X}(\mathcal{O}_{K'}), \\ A \text{ constant on } B(P, e(n+1)) \\ \text{where } e = e(K'/K) \end{smallmatrix} \right\}$$

$$\mathrm{Ev}_{-1} \,\mathrm{Br}X = \left\{ A \in \mathrm{Br}X \;:\; \begin{smallmatrix} \forall K'/K \text{ finite } \forall P \in \mathcal{X}(\mathcal{O}_{K'}), \\ A \text{ constant on } \mathcal{X}(\mathcal{O}_{K'}) \end{smallmatrix} \right\}$$

$$\mathrm{Ev}_{-2} \,\mathrm{Br}X = \left\{ A \in \mathrm{Br}X \;:\; \begin{smallmatrix} \forall K'/K \text{ finite } \forall P \in \mathcal{X}(\mathcal{O}_{K'}), \\ A \text{ is zero on } \mathcal{X}(\mathcal{O}_{K'}) \end{smallmatrix} \right\}$$

Now let $K = \mathrm{Frac}(\mathcal{O}^h_{\mathcal{X},Y})$ and $F = \mathbb{F}(Y)$, then Kato defined $\mathrm{fil}_n \,\mathrm{Br}K$ and injections

$$\mathrm{rsw}_n : \frac{\mathrm{fil}_n \,\mathrm{Br}(K)}{\mathrm{fil}_{n-1} \,\mathrm{Br}(K)} \to \mathfrak{m}^{-n} \otimes \omega_F^2 \cong \Omega_F^2 \oplus \Omega_F^1$$

which is an element of the form $\pi^{-n}(\alpha + \beta \frac{d\pi}{\pi})$. It turns out that Kato's filtration is not quite the right thing to do here.

Definition 21. For $n \geq 1$

$$\widetilde{\mathrm{fil}}_n \mathrm{Br}X := \left\{ A \in \mathrm{fil}_{n+1} \,\mathrm{Br}X \;:\; \mathrm{rsw}_n(A) \in \Omega_F^2 \right\}$$

Remark 22. $n\alpha = d\beta$, so $\widetilde{\mathrm{fil}}_n = \mathrm{fil}_n$ if $p \nmid (n+1)$

Theorem 23 (Theorem A)**.** *For* $n \geq 1$

$$\begin{aligned} \mathrm{Ev}_n \,\mathrm{Br}X &= \widetilde{\mathrm{fil}}_n \mathrm{Br}X \\ \mathrm{Ev}_0 \,\mathrm{Br}X &= \mathrm{fil}_0 \,\mathrm{Br}X \\ \mathrm{Ev}_{-1} \,\mathrm{Br}X &= \left\{ A \in \mathrm{fil}_0 \,\mathrm{Br}X \;:\; \partial A \in H^1(\mathbb{F}.\mathbb{Q}/\mathbb{Z}) \subseteq H^q(F, \mathbb{Q}/\mathbb{Z}) \right\} \\ \mathrm{Ev}_{-2} \,\mathrm{Br}X &= \left\{ A \in \mathrm{fil}_0 \,\mathrm{Br}X \;:\; \partial A = 0 \in H^1(\mathbb{F}.\mathbb{Q}/\mathbb{Z}) \right\} \end{aligned}$$

Remark 24. This is all happening in mixed characteristic, for the equal characteristic version see the very recent work of A. Krishna and S. Majumder.

We prove this through a local study of the evaluation map.

Recall that for $P_n \in \mathcal{X}(\mathcal{O}_K/\pi^n)$, lifts to $\mathcal{X}(\mathcal{O}_K/\pi^{n+1})$ are parametrised precisely by the tangent space $T_{P_0}Y$. We get for $P \in \mathcal{X}(\mathcal{O}_K)$ and $Q \in B(P, n)$, a vector $[P, Q]_n \in T_{P_0}Y$ identifying Q modulo π^{n+1}.

Theorem 25 (Theorem B)**.** *Let* $A \in \mathrm{fil}_n \,\mathrm{Br}X$ *for* $n > 0$ *and write*

$$\mathrm{rsw}_n(A) = (\alpha, \beta).$$

Then α, β *are regular on* Y*. For* $P \in \mathcal{X}(\mathcal{O}_K)$ *and* $Q \in B(P, n)$*,*

$$\mathrm{inv}(A(Q)) = \mathrm{inv}(A(P)) + \frac{1}{p}\mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(\beta_{P_0}([P, Q]_n)).$$

In particular, A *is constant on* $B(P, n+1)$*, and if* $\beta_{P_0} \neq 0$ *then* A *is nonconstant on* $B(P, n)$*.*

Remark 26. This is somehow the obvious thing to do: β was a 1-form and $[P, Q]$ was a tangent vector, so of course we should localise and evaluate. But that gave us an element of $\mathbb{F}$, and that's not obviously in $\mathbb{Q}/\mathbb{Z}$... so surely we should take the trace down to $\mathbb{F}_p$ which then has a natural map to $\mathbb{Q}/\mathbb{Z}$ by lifting to the integers and multiplying by $1/p$.

We claim that Theorem B implies Theorem A. Indeed the proof goes as follows.

- Prove for fil_0 using the same argument as in the prime-to-p part
- use Theorem B to make the inductive step from fil_{n-1} to fil_n.

Corollary 27 (Corollary of Theorem A). *The following hold*

- *If* $H^0(Y, \Omega^1) = H^0(Y, \Omega^2) = 0$, *then* $\mathrm{fil}_0 \operatorname{Br} X = \operatorname{Br} X$.
- *If* $H^0(Y, \Omega^1) = 0$ *and* $e(K) < p - 1$, *then* $\mathrm{fil}_0 \operatorname{Br} X = \operatorname{Br} X$.
- *If* $H^1(Y, \mathbb{Q}/\mathbb{Z}) = 0$ *then* $\mathrm{fil}_0 \operatorname{Br} X = \mathrm{Ev}_{-1} \operatorname{Br} X$.

In particular, for a smooth projective variety V over a number field, with $\mathrm{Pic}(\overline{V})$ *finitely generated and torsion free, the last two hold for almost all primes.*

Theorem 28 (Theorem D). *Let V be a smooth proper geometrically integral variety over a number field L with $H^0(V, \Omega^2) \neq 0$. Let $\mathfrak{p} | p$ be a prime of good ordinary reduction. Then after extending L, there exists $A \in \operatorname{Br} V[p^\infty]$ such that evaluation is nonconstant. In particular, V fails weak approximation.*

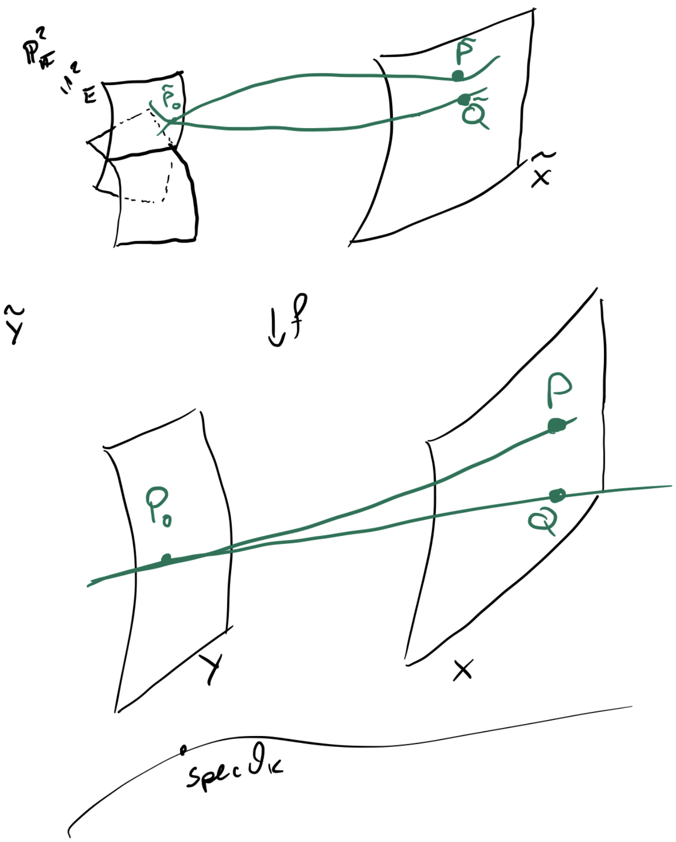
Proof. If $H^q(Y, B^r) = 0$ for all q, r (here B is as in the de Rham complex from the first lecture), we say this is the ordinary case. Then Bloch–Kato use the Hodge decomposition in this case to get

$$\mathrm{fil}_0 H^2(\overline{V}, \mathbb{Q}_p) \text{``}=\text{''}\, H^{0,2} \oplus H^{1,1}$$

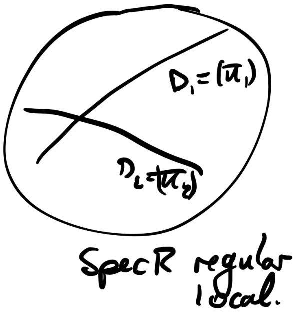
If $H^{2,0}$ is nonzero then there is an element outside of fil_0.

See the paper of Ambrosi–Newton–Pagano (2025) where they were able to replace the étale cohomology here with prismatic cohomology and so get away from the ordinary assumption. □

Okay, so what about the proof of Theorem B?



Proof sketch for Theorem B.



The problem is to relate $\mathrm{rsw}_E(f^*A)$ to $\mathrm{rsw}_Y(A)$ via $\mathrm{rsw}_{\tilde{Y}}(A)$.
$\chi \in H^q(\mathrm{Spec}(R)\backslash D_1 \cup D_2)$, write $\mathrm{rsw}_{D_i}(\chi) = \pi_1^{-n_1}\pi_2^{-n_2}(\alpha_i + \beta_i \frac{d\pi_i}{\pi_i})$, where here $n_i = \mathrm{sw}_{D_i}(\chi)$, then $\alpha_i, \beta_i \in \Omega^{q,q-1}_{R/\pi_i}(\log D_j)$ □

Lecture 4 (Molyakov): Introduction to Prismatic Cohomology

3. Motivation

Assume we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & Y \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec}(\mathbb{Q}) \arrow[r] & \operatorname{Spec}(\mathbb{Z}_{(p)}) & \operatorname{Spec}(\mathbb{F}_p) \arrow[l]
\end{tikzcd}$$

Where we have good reduction mod p. Then for $\ell \neq p$ we have $H^n_{\text{et}}(\overline{X}, \mathbb{Z}_\ell) \cong H^n_{\text{et}}(\overline{Y}, \mathbb{Z}_\ell)$ and indeed this is also $H^n(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. The coprime-to-$p$ torsion of $H^n(X_{\mathbb{C}}, \mathbb{Z})$ is determined by Y.

What about ℓ-torsion for $\ell = p$?

Example 29 (Bhatt–Morrow–Scholze 2019)**.** There exist two smooth projective 3-folds X_1, X_2 with isomorphic good reductions mod p, and

$$H^1((X_1)_{\mathbb{C}}, \mathbb{Z}/p) = (\mathbb{Z}/p\mathbb{Z})^2$$
$$H^1((X_2)_{\mathbb{C}}, \mathbb{Z}/p) = (\mathbb{Z}/p\mathbb{Z})^3$$

So the p-torsion is not determined by the special fibre alone!

Theorem 30 (Bhatt–Morrow–Scholze)**.** *Let R be a DVR of mixed characteristic $(0,p)$ and $K = \operatorname{Frac}(R)$, $k = R/\mathfrak{m}$. Let $\mathcal{X}/R$ be smooth and proper, so that we have*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & Y \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec}(K) \arrow[r] & \operatorname{Spec}(R) & \operatorname{Spec}(k) \arrow[l]
\end{tikzcd}$$

we have $\dim_{\mathbb{F}_p} H^n(\overline{X}, \mathbb{Z}/p\mathbb{Z}) \leq \dim_k H^n_{dR}(Y)$

Remark 31. $H^n_{dR}(Y) = H^n(\Omega^\bullet_{Y/k})$, and we have the usual spectral sequence

$$H^s(Y, \Omega^t_Y) \implies H^n(\Omega^\bullet_{Y/k})$$

with $n = s + t$. Moreover

$$\dim_k H^n_{dR}(Y) \leq \sum_{s+t=n} \dim_k H^s(Y, \Omega^t)$$

Usual Hodge theory tells us that for smooth proper $X/\mathbb{C}$,

$$H^n(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{s+t=n} H^s(X, \Omega^t)$$

and we'd like a p-adic analogue. Consider $\mathbb{C}_p = \overline{\mathbb{Q}_p}^\wedge$, then there is an analogue.

Theorem 32 (Faltings)**.** *Let K be a complete discretely valued field of characteristic 0 and residue characteristic $p > 0$. Assume that the residue field is perfect. Let $\mathbb{C}_p = \overline{K}^\wedge$. Then for X/K smooth and proper we have*

$$H^n_{\text{et}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \bigoplus_{s+t=n} H^s(X, \Omega^t) \otimes_K \mathbb{C}_p(-t)$$

which is an isomorphism of Γ_K-modules.

Theorem 33. *Let X be a smooth proper variety over $\mathbb{C}$. Then for every Zariski open $U \subseteq X$ we have* $\dim \operatorname{im}\left(H^n(X, \mathbb{Z}/p\mathbb{Z}) \to H^n(U, \mathbb{Z}/p\mathbb{Z})\right) \geq h_X^{0,n}$*, where*

$$h_X^{0,n} = \dim_{\mathbb{C}} H^n(X, \mathcal{O}_X).$$

Prismatic cohomology should somehow see crystalline, de Rham, Hodge–Tate, and p-adic étale cohomology all at once.

4. δ-RINGS

Let A be a ring without p torsion and assume that we have $\phi : A \to A$ a ring homomorphism lift of the Frobenius $\mathrm{Fr} : A/p \to A/p$ and write

$$\phi(x) = x^p + p\delta(x).$$

Since $\phi(xy) = \phi(x)\phi(y)$, we can deduce from evaluating both sides that

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y). \tag{1}$$

Moreover since $\phi(x + y) = \phi(x) + \phi(y)$ we get

$$\delta(x + y) = \frac{x^p + y^p - (x + y)^p}{p} + \delta(x) + \delta(y). \tag{2}$$

Definition 34. A δ-ring A is a pair (A, δ) of a ring A and a map $\delta : A \to A$ satisfying (1)(2) and such that $\delta(1) = \delta(0) = 0$.

Then there is a map

$$\{\delta\text{-structures}\} \to \{\text{lifts of Fr}\}$$
$$\delta \mapsto \phi(x) = x^p + p\delta(x)$$

which is bijective when A is torsionfree.

Remark 35. δ-structures also correspond to sections of the map we've discussed before $W_2(A) \to A$.

Example 36. Consider $A = \mathbb{Z}$. Then ϕ must be the identity map and so the only δ-structure is $\delta(n) = \frac{n - n^p}{p}$.

Moreover, the forgetful functor from δ-rings to rings has both adjoints: left and right.

$$R(A) := W(A)$$
$$L(\mathbb{Z}[x]) := \mathbb{Z}\{x\} = \mathbb{Z}[x, \delta(x), \delta^2(x), \dots]$$

where $\delta(\delta^n(x)) = \delta^{n+1}(x)$.

5. PRISMS

Okay, now let us pass on to prisms! A prism will correspond to a δ-ring and an ideal.

Definition 37. A prism is a pair (A, I) where A is a δ-ring and $I \leq A$ is an invertible (i.e. locally principal) ideal such that the following properties are satisfied:

1. A is (I, p)-complete;
2. $p \in I + \phi(I) \cdot A$ (for ϕ the frobenius associated to our δ-ring A);
3. A has bounded p-torsion ($A[p^\infty] = A[p^N]$ for some N).

From now on we'll always denote by $\phi : A \to A$ the lift of Frobenius corresponding to the δ-structure.

Remark 38. Consider the closed subscheme in $\mathrm{Spec}(A)$ given by the ideal I and the one by $\phi(I)$. This is saying the intersection is of characteristic p.

Definition 39. We say that a prism (A, I) is

- perfect if $\phi : A \to A$ is an isomorphism.
- crystalline if $I = \langle p \rangle$

Example 40. $\mathbb{Z}_p$ with the trivial δ-structure is perfect (Frobenius lifts to the identity map), and we can choose $I = \langle p \rangle$ for the ideal to get that $(\mathbb{Z}, \langle p \rangle)$ is a perfect crystalline prism.

Example 41. Another perfect crystalline prism is of the form $(W(R), \langle p \rangle)$ where R is a perfect $\mathbb{F}_p$-algebra.

Example 42. Consider $(\mathbb{Z}_p[[u]], (u-p))$ with the unique δ-structure such that the lift of Frobenius sends $\phi(u) = u^p$. We claim that it is a prism which is neither complete nor crystalline. Indeed, it is a prism $\langle I, \phi(I) \rangle = \langle u-p, u^p - p \rangle$ which contains p. ϕ is clearly not an isomorphism and nor is $I = \langle p \rangle$.

Lemma 43 (rigidity)**.** *Let $f : (A, I) \to (B, J)$ be a map of prisms (i.e. a ring homomorphism such that $f(I) \subseteq J$ and $f = \circ\delta_A = \delta_B \circ f$). Then*

$$f(I) = J.$$

6. Perfectoid rings

Let

$$R^\flat = \lim_{\leftarrow_\phi} R/p = (\dots \xrightarrow{\phi} R/p \xrightarrow{\phi} R/p)$$

Definition 44. Let $A_{\mathrm{inf}}(R) = W(R^\flat)$

Example 45. For $R = \mathbb{Z}$, $R^\flat = \mathbb{Z}/p\mathbb{Z}$.

Lemma 46. *For R p-complete, we have an isomorphism*

$$\lim_{x \mapsto x^p} R \to R^\flat.$$

In particular we obtain maps

$$\psi : R^\flat \to R$$

For R a p-complete ring, we then have maps

$$A_{\mathrm{inf}}(R) \to R$$

induced as follows.

$$W(R^\flat) \to R$$
$$\sum p^i [x_i] \mapsto \sum p^i \psi(x_i)$$

with commutative diagram

$$\begin{tikzcd}
A_{\mathrm{inf}}(R) \arrow[r, "\theta"] \arrow[d] & R \arrow[d] \\
R^\flat \arrow[r] & R/p
\end{tikzcd}$$

Definition 47. A perfectoid ring is a ring of the form A/I for a perfect prism (A, I).

We get natural functions

$$\begin{aligned} \{\text{perfectoid rings}\} &\leftrightarrow \{\text{perfect prisms}\} \\ A/I &\leftarrow (A, I) \\ R &\mapsto (A_{\text{inf}}(R), \ker(\theta)) \end{aligned}$$

Definition 48. Let (A, I) be a base prism and $\overline{A} = A/I$. Let $\mathcal{X}$ be a p-formal scheme over $\overline{A}$. Then the prismatic site $(\mathcal{X}/A)_{\Delta}$ has

- objects: a map of prisms $(A, I) \to (B, IB)$ together with a map $\text{Spf}(B/IB) \to \mathcal{X}$ such that the diagram below commutes.

$$\begin{tikzcd}
\text{Spf}(B/IB) \arrow[r] \arrow[d] & \text{Spf}(B) \arrow[dd] \\
\mathcal{X} \arrow[d] & \\
\text{Spf}(\overline{A}) \arrow[r] & \text{Spf}(A)
\end{tikzcd}$$

- morphisms: morphisms such that the diagram below commutes

$$\begin{tikzcd}
& \text{Spf}(B/IB) \arrow[r] \arrow[dl] \arrow[dd] & \text{Spf}(B) \arrow[dd] \\
\mathcal{X} & & \\
& \text{Spf}(B'/IB') \arrow[r] \arrow[ul] & \text{Spf}(B')
\end{tikzcd}$$

- covers: $(\mathcal{X} \leftarrow \text{Spf}(B) \to \text{Spf}(B/IB))$ with $\bigcup_{\alpha} \text{Spf}(B_{\alpha}) \to \text{Spf}(B)$ flat (Zariski)

Remark 49. If $\mathcal{X}$ is affine then can replace Spf with Spec and take the trivial topology.

Definition 50. Let

$$\mathcal{O}_{\Delta}(\mathcal{X} \leftarrow \text{Spf}(B) \to \text{Spf}(B/IB)) = B.$$

Then take

$$R\Gamma_{\Delta}(\mathcal{X}/A) = R\Gamma((\mathcal{X}/A)_{\Delta}, \mathcal{O}_{\Delta}) = \text{RHom}(\mathbb{Z}_{\Delta}, \mathcal{O}_{\Delta}) = \text{Rlim}_{T \in (\mathcal{X}/A)_{\Delta}} \mathcal{O}_{\Delta}(T).$$

Indeed, since B all have compatible δ-structures in a cover, we get that $R\Gamma_{\Delta}(\mathcal{X}/A)$ is a (p, I)-complete A algebra object a lift of Frobenius

$$\phi_{\Delta} : R\Gamma_{\Delta}(\mathcal{X}/A) \to \phi_* R\Gamma_{\Delta}(\mathcal{X}/A)$$

Theorem 51 (Bhatt-Scholze)**.** *Let (A, I) be a prism and $\mathcal{X}$ be a smooth formal scheme over $\overline{A}$.*

(1) If $(A, I) = (W(k), (p))$ where k is a perfect field of characteristic p, then

$$\phi_*(R\Gamma_{\Delta}(\mathcal{X}/A)) \cong R\Gamma_{\text{crys}}(\mathcal{X}/W(k)).$$

(2) If $\mathcal{X} = \mathrm{Spf}(R)$ *is affine then*

$$R\Gamma_{\Delta}(\mathcal{X}/A) \otimes_A^{\mathbb{L}} \overline{A} \cong \Omega^q_{R/\overline{A}} \otimes (I/I^2)^{\otimes -q}.$$

(3)

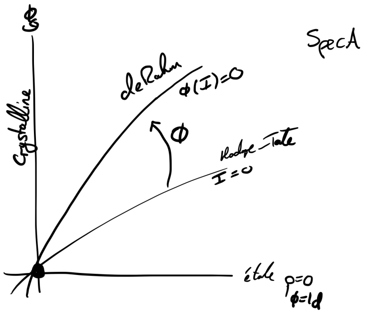
$$\phi^* R\Gamma_{\Delta}(\mathcal{X}/A) \otimes_A^{\mathbb{L}} \overline{A} \cong R\Gamma(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}}).$$

(4) Let (A, I) *be locally perfect. Then*

$$\left(R\Gamma_{\Delta}(\mathcal{X}/A)[\tfrac{1}{I}] \otimes_A A/p^n\right)^{\phi_{\Delta}=1} \cong R\Gamma_{\mathrm{et}}(\mathcal{X}_{\eta}, \mathbb{Z}/p^n\mathbb{Z})$$

Remark 52. If $\mathcal{X}$ is smooth and proper over $\overline{A}$ then all cohomology groups are the usual ones

This information is best viewed through the diagram below. In particular Frobenius relates Hodge–Tate and de Rham cohomology.



7. Application

Let $\mathbb{C}_p = \overline{\mathbb{Q}_p}^{\wedge}$ and take

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & Y \arrow[l] \arrow[d] \\
\mathrm{Spec}(\mathbb{C}_p) \arrow[r] & \mathrm{Spec}(\mathcal{O}_{\mathbb{C}_p}) & \mathrm{Spec}(k) \arrow[l]
\end{tikzcd}$$

$$\dim_{\mathbb{F}_p} H^n(\overline{X}, \mathbb{Z}/p) \leq \dim_k H^n_{dR}(Y).$$

Fortunately, $\mathcal{O}_{\mathbb{C}_p}$ is already perfectoid. Consider the perfect prism induced by the top row of the diagram below.

$$\begin{tikzcd}
A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p}) = W(\mathcal{O}_{\mathbb{C}_p^{\flat}}) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{C}_p} \arrow[d] \\
\mathcal{O}^{\flat}_{\mathbb{C}_p} \arrow[r] & \mathcal{O}_{\mathbb{C}_p}/p.
\end{tikzcd}$$

$$R\Gamma_{\Delta}(\mathcal{X}/A) = R\Gamma(\mathcal{X}/A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p})).$$

Then

$$\left(R\Gamma_{\Delta}(\mathcal{X}/A) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} \mathbb{C}_p^{\flat}\right)^{\phi=1} = R\Gamma_{\mathrm{et}}(\overline{X}, \mathbb{Z}/p).$$

In fact it turns out that

$$\begin{aligned}
H^n_{\mathrm{et}}(\overline{X}, \mathbb{Z}/p\mathbb{Z}) &= \dim_{\mathbb{F}_p} H^n\left(\left(R\Gamma_\Delta(\mathcal{X}/A) \otimes \mathbb{C}_p^\flat\right)^{\phi=1}\right) \\
&\leq \dim_{\mathbb{C}_p} H^n\left(R\Gamma_\Delta(\mathcal{X}/A) \otimes \mathbb{C}_p^\flat\right) \\
&\leq \dim_k H^n\left(R\Gamma_\Delta(\mathcal{X}/A) \otimes k\right) \\
&= \dim_k H^n_{dR}(Y)
\end{aligned}$$