# Supplementary sheet on Galois cohomology

Ross Paterson and Happy Uppal

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## 1 Quick Reference

The following is a quick guide to some results/notation in group/Galois cohomology.
**We have starred all the things which are vital for the lectures**.

We fix $G$ to be a finite group and $M$ a $G$-module.

| **Object** | **Formula / statement** | **Section** |
|---|---|---|
| *$\mathrm{H}^0(G,M)$ | $M^G$ (fixed points) or $(M)^{\sigma=1}$ if $G=\langle\sigma\rangle$ | §4.1 |
| *$Z^1(G,M)$ | maps $f: G \to M$ with $f(g_1g_2)=f(g_1)+g_1f(g_2)$ | §4.1 |
| *$B^1(G,M)$ | maps $f: G \to M$ with $f(g)=g\cdot m - m$ for fixed $m \in M$ | § 4.1 |
| $G$ acts trivially on $M$ | $\mathrm{H}^1(G,M)=\mathrm{Hom}(G,M)$ | Ex. 4.7. |
| *$G=\langle\sigma\rangle$ is cyclic | $\mathrm{H}^0(G,M)=M^G$ | Prop 4.13 |
| | $\mathrm{H}^{2n+1}(G,M)=\ker \mathbf{N}/\mathrm{im}(1-\sigma)$ | |
| | $\mathrm{H}^{2n+2}(G,M)=\ker(1-\sigma)/\mathrm{im}\,\mathbf{N}$ | |
| *Hilbert 90 | $\mathrm{H}^1(\mathrm{Gal}(L/K),L^\times)=0$ | Thm 4.10 |
| *Inflation map | $\mathrm{Inf}: \mathrm{H}^n(G/H,M^H)\to\mathrm{H}^n(G,M)$ | Def. 5.2 |
| *Restriction map | $\mathrm{Res}: \mathrm{H}^n(G,M)\to\mathrm{H}^n(H,M)$ | Def. 5.1 |
| Inf-Res sequence | $0\to\mathrm{H}^1(G/H,M^H)\xrightarrow{\mathrm{Inf}}\mathrm{H}^1(G,M)\xrightarrow{\mathrm{Res}}\mathrm{H}^1(H,M)$ | Thm. 5.3 |
| *Cup product | $\smile: \mathrm{H}^p(G,M)\times\mathrm{H}^q(G,N)\to\mathrm{H}^{p+q}(G,M\otimes_{\mathbb{Z}}N)$ | Def 7.1. |
| Shapiro's Lemma | $\mathrm{H}^n(G,\mathrm{Ind}_H^G N)\cong\mathrm{H}^n(H,N)$ | Thm A.6 |
| Herbrand quotient | $h^{2/1}(M)=\lvert\mathrm{H}^2(G,M)\rvert/\lvert\mathrm{H}^1(G,M)\rvert$ | §B |
| | (or $h_{2/1}$ in Emerton's sheet) | |

For a field $K$, we denote by $G_K$ the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

## 2 $G$-Modules

**Definition 2.1.** A left $G$-module $M$ is an abelian group $(M,+)$ on which $G$ acts such that

1. $\mathrm{id}_G m = m$ for all $m \in M$,
2. $g(m_1+m_2)=gm_1+gm_2$ for all $g\in G$ and $m_1,m_2\in M$,
3. $(g_1g_2)m=g_1(g_2m)$ for all $g_1,g_2\in G$ and $m\in M$.

### 2.1 Examples

**Example 2.2.** Let $G$ be any group and $M$ any abelian group. Then we can always endow $M$ with a $G$-module structure by defining $g \cdot m = m$ for all $g \in G$ and $m \in M$. In this case we call $M$ a *trivial* $G$-module.

**Example 2.3.** Let $L/K$ be Galois with $G = \mathrm{Gal}(L/K)$. Then $(L, +)$ (resp. $(L^\times, \times)$) are a $G$-modules via $\sigma \cdot x = \sigma(x)$ for all $\sigma \in G$ and $x \in L$ (resp. $L^\times$).

## 3 $G$-module homomorphisms

**Definition 3.1.** Let $M$ be a $G$-module. A *submodule* of $M$ is a subgroup $N \subseteq M$ which is also a $G$-module.

**Definition 3.2.** A *$G$-module homomorphism* $f : M \to N$ is a group homomorphism with $f(g \cdot m) = g \cdot f(m)$ for all $g \in G$, $m \in M$. We write $\mathrm{Hom}_G(M, N)$ for the abelian group of such maps.

**Exercise 3.3.** Show that the kernel and image of $G$-module homomorphism $f : M \to N$ are submodules of $M$ and $N$ respectively.

## 4 Cohomology via explicit inhomogenous cochains

**Definition 4.1.** Let $M$ be a $G$-module. For $n \geq 0$, set

$$C^n(G, M) := \{\text{maps } G^n \to M\}.$$

Elements of $C^n(G, M)$ are called *$n$-cochains*. The *coboundary maps* $d^n : C^n(G, M) \to C^{n+1}(G, M)$ are

$$\begin{aligned}(d^n f)(g_1, \dots, g_{n+1}) = g_1 \cdot f(g_2, \dots, g_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \dots, g_n).\end{aligned}$$

*Remark* 4.2. When $G$ is profinite and $M$ has the discrete topology, we require the maps in $C^n(G, M)$ to be continuos.

**Exercise 4.3.** Show that $d^{n+1} \circ d^n = 0$.

**Definition 4.4.** For a $G$ module $M$, the group of *$n$-cocycles* is

$$Z^n(G, M) := \ker\left(d^n : C^n(G, M) \to C^{n+1}(G, M)\right)$$

and the group of *$n$-coboundaries* is

$$B^n(G, M) := \mathrm{im}\left(d^{n-1} : C^{n-1}(G, M) \to C^n(G, M)\right).$$

**Definition 4.5.** Let $M$ be a $G$-module. The *$n$-th cohomology group* of $M$ is

$$\mathrm{H}^n(G, M) := \frac{Z^n(G, M)}{B^n(G, M)}.$$

**Lemma 4.6.** *Given a short exact sequence of $G$-modules*

$$0 \to M' \to M \to M'' \to 0$$

*there exists a canonical long exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(G,M') \arrow[r] & \mathrm{H}^0(G,M) \arrow[r] & \mathrm{H}^0(G,M'') \arrow[dll, "\delta^0"] & \\
 & \mathrm{H}^1(G,M') \arrow[r] & \mathrm{H}^1(G,M) \arrow[r] & \mathrm{H}^1(G,M'') \arrow[dll, "\delta^1"] & \\
 & \mathrm{H}^2(G,M') \arrow[r] & \mathrm{H}^2(G,M) \arrow[r] & \mathrm{H}^2(G,M'') \arrow[dll, dashed] & \\
 & \mathrm{H}^n(G,M') \arrow[r] & \mathrm{H}^n(G,M) \arrow[r] & \mathrm{H}^n(G,M'') \arrow[r] & \dots
\end{tikzcd}$$

*in which the $\delta^i$ are certain connecting homomorphisms*

## 4.1 Low-degree cochains

**Degree 0:** $(d^0 m)(g) = g \cdot m - m$, so

$$\boxed{\mathrm{H}^0(G,M) = M^G = \{m \in M : gm = m \text{ for all } g \in G\}}.$$

**Degree 1:** A 1-cochain is a map $f : G \to M$. Then $(d^1 f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1)$, so

$$\boxed{Z^1(G,M) = \{f : G \to M : f(g_1 g_2) = f(g_1) + g_1 \cdot f(g_2) \text{ for all } g_1, g_2 \in G.\}}$$

The 1-coboundaries are

$$\boxed{B^1(G,M) = \{f : G \to M : f(g) = g \cdot m - m \text{ for fixed } m \in M\}.}$$

**Exercise 4.7.** Show that when $M$ is a trivial $G$-module, then $Z^1(G,M) = \mathrm{Hom}(G,M)$ and $B^1(G,M) = 0$, so $\mathrm{H}^1(G,M) = \mathrm{Hom}(G,M)$. Note that if $G$ is profinite, these are continuos homomorphisms.

**Example 4.8.** An example of the above exercise would be $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ acting on $M = \mu_2 = \{\pm 1\}$. Then

$$\mathrm{H}^1(G_K, \mu_2) = \mathrm{Hom}(G_K, \mu_2),$$

the group of continuous characters $G_K \to \{\pm 1\}$.

## 4.2 Hilbert 90

An important result in group cohomology is Hilbert 90.

**Lemma 4.9** (Linear independence of characters)**.** *Let $G$ be a group and $L$ a field. Let $\chi_1, \dots, \chi_n : G \to L$ be distinct homomorphisms. Then $\chi_1, \dots, \chi_n$ are linearly independent, that is if there exists $\lambda_1, \dots, \lambda_n \in L$ such that*

$$\sum_i \lambda_i \chi_i = 0$$

*then $\lambda_i = 0$ for all $i$.*

**Theorem 4.10** (Hilbert 90)**.** *Let $L/K$ be a finite Galois extension with Galois group $G$. Then*

$$\mathrm{H}^1(G, L^\times) = 0.$$

*Proof.* Let $\phi \in Z(G, L^\times)$ i.e. $\phi(\tau\sigma) = \phi(\tau)\tau(\phi(\sigma))$ for $\sigma, \tau \in G$. We want to show there exists $a \in L^\times$ such that $\phi(\tau) = \tau(a)a^{-1}$. Note that each $\sigma \in G$ is a character $\sigma : L^\times \to L$ and the collection of elements are distinct. We have by Lemma 4.9 that there exists $c \in L^\times$ such that

$$b := \sum_{\sigma \in G} \phi(\sigma)\sigma(c) \neq 0.$$

For any $\tau \in G$ we have

$$\phi(\tau)\tau(b) = \phi(\tau) \sum_{\sigma \in G} \tau(\phi(\sigma)\sigma(c)) = \sum_{\sigma \in G} \phi(\tau)\tau(\phi(\sigma))\tau(\sigma(c)) = \sum_{\sigma \in G} \phi(\tau\sigma)(\tau\sigma)(c) = b$$

Hence, $\phi(\tau) = b\tau(b)^{-1}$. Let $a := b^{-1}$ and we have found our $a$. □

*Remark* 4.11. From this finite case we can deduce

$$\mathrm{H}^1(G_K, (K^{\mathrm{sep}})^\times) = 0$$

*Remark* 4.12. Note that the classical version of Hilbert 90 is as follows. Let $L/K$ be a cyclic Galois extension and $G := \mathrm{Gal}(L/K)$ generated by $\sigma$. Then there exists $a \in L^\times$ such that $\mathrm{Norm}_{L/K}(a) = 1$ if and only if there exists $b \in L^\times$ such that

$$a = \frac{\sigma(b)}{b}.$$

## 4.3 Cohomology of Cyclic Groups

Let $G = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}$ and $M$ a $G$-module. Write $\mathbf{N} = 1 + \sigma + \cdots + \sigma^{n-1}$ and $D = \sigma - 1$.

**Proposition 4.13.** *We have*

$$\mathrm{H}^n(G, M) = \begin{cases} M^G & \text{if } n = 0, \\ \ker \boldsymbol{N}/\mathrm{im}(1-\sigma) & \text{if } n \text{ is odd}, \\ \ker(1-\sigma)/\mathrm{im}\,\boldsymbol{N} & \text{if } n \text{ is even and non-zero.} \end{cases}$$

**Exercise 4.14.** Show that the values of $\mathrm{H}^n(G, M)$ in Proposition 4.13 are consistent with the cohchain definition of group cohomology for $n = 0, 1$ and 2.

**Exercise 4.15.** Let $G = \mathbb{Z}/n\mathbb{Z}$ act trivially on $M = \mathbb{Z}$ and compute $\mathrm{H}^n(G, M)$ for $n \geq 0$.

**Exercise 4.16.** Let $G = \mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$ act on $M = \mathbb{Z}$ by $\sigma \cdot n = -n$. Compute $\mathrm{H}^n(G, M)$ for $n \geq 0$.

# 5 Inflation and Restriction

Let $H \trianglelefteq G$ be a normal subgroup and $M$ a $G$-module. Since $H$ acts on $M$, the subgroup $M^H$ of $H$-fixed points is a $G/H$-module (via $\bar{g} \cdot m = g \cdot m$ for any lift $g \in G$).

## 5.1 Restriction

**Definition 5.1** (Restriction)**.** The *restriction map* $\mathrm{Res} : \mathrm{H}^n(G, M) \to \mathrm{H}^n(H, M)$ is induced by pre-composing cochains with the inclusion $H \hookrightarrow G$:

$$\mathrm{Res}(f)(h_1, \ldots, h_n) := f(h_1, \ldots, h_n).$$

This is defined for any subgroup $H \leq G$ (not just normal ones).

### 5.2 Inflation

**Definition 5.2** (Inflation)**.** The *inflation map* $\mathrm{Inf} : \mathrm{H}^n(G/H, M^H) \to \mathrm{H}^n(G, M)$ is induced by precomposing cochains with the quotient $\pi : G \to G/H$:

$$\mathrm{Inf}(f)(g_1, \dots, g_n) := f(\bar{g}_1, \dots, \bar{g}_n) \quad \text{where } \bar{g}_i = \pi(g_i) \in G/H.$$

Since $f$ takes values in $M^H \subset M$, this is a well-defined $G$-cochain.

### 5.3 The inflation-restriction exact sequence

**Theorem 5.3** (Inflation-restriction exact sequence)**.** *Let $A$ be a $G$-module and $H$ a normal subgroup of $G$. Then there is an exact sequence*

$$0 \to \mathrm{H}^1(G/H, A^H) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(H, A).$$

**Example 5.4.** Let $K \subset L \subset F$ be Galois with $H = \mathrm{Gal}(F/L) \trianglelefteq G = \mathrm{Gal}(F/K)$. Then inflation-restriction gives

$$0 \to \mathrm{H}^1(\mathrm{Gal}(L/K), M^H) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(G_K, M) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(G_L, M).$$

A class in $\mathrm{H}^1(G_K, M)$ restricting to zero over $L$ comes from a class over $\mathrm{Gal}(L/K)$.

## 6 Kummer theory

**Definition 6.1** (Kummer extension)**.** Let $K$ be a field with $\mathrm{char}(K) \nmid n$, and suppose $\mu_n \subset K$. A *Kummer extension* of $K$ of exponent dividing $n$ is a finite Galois extension $L/K$ with $\mathrm{Gal}(L/K)$ abelian of exponent dividing $n$.

**Theorem 6.2** (Kummer correspondence)**.** *Assume $\mu_n \subset K$. There is a perfect pairing of finite abelian groups*

$$\mathrm{Gal}(L/K) \times \Delta_L \longrightarrow \mu_n, \qquad (\sigma, a) \mapsto \frac{\sigma(\alpha)}{\alpha},$$

*where $\Delta_L = \{a \in K^\times : a^{1/n} \in L\}/(K^\times)^n$ and $\alpha$ is any $n$-th root of $a$ in $L$. This gives a natural bijection*

$$\left\{ \textit{subgroups } \Delta \subset K^\times/(K^\times)^n \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{abelian extensions } L/K \\ \textit{of exponent dividing } n \end{array} \right\}$$

$$\Delta \longmapsto K(\Delta^{1/n}) := K\big(\{a^{1/n} : a \in \Delta\}\big)$$

*with* $|\mathrm{Gal}(L/K)| = [L : K] = |\Delta_L|$.

**Exercise 6.3.** Let $K$ be a field of characteristic coprime to 2 and $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

(a) Apply the long exact sequence to $1 \to \mu_2 \to \overline{K}^\times \xrightarrow{x \mapsto x^2} \overline{K}^\times \to 1$ and use Hilbert 90 to show $\mathrm{H}^1(G_K, \mu_2) \cong K^\times/(K^\times)^2$.

(b) For $K = \mathbb{Q}$: show $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is an $\mathbb{F}_2$-vector space with basis $\{-1, p : p \text{ prime}\}$, and for $d \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ describe the corresponding characters $G_\mathbb{Q} \to \mu_2$ (i.e. the element of $\mathrm{H}^1(G_\mathbb{Q}, \mu_2) = \mathrm{Hom}(G_\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$).

**Exercise 6.4.** Let $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{2})$, $G = G_\mathbb{Q}$, $H = G_L$, $M = \mu_2$ (trivial action).

(a) Show $\mathrm{Inf} : \mathrm{H}^1(\mathrm{Gal}(L/K), \mu_2) \to \mathrm{H}^1(G_\mathbb{Q}, \mu_2)$ is injective.

(b) Show a character $\chi : G_\mathbb{Q} \to \mu_2$ is in the image of Inf if and only if $\chi|_{G_L} = 1$. Which squarefree integers $d$ give such $\chi_d$?

(c) Compute $\mathrm{Res} : \mathrm{H}^1(G_\mathbb{Q}, \mu_2) \to \mathrm{H}^1(G_L, \mu_2)$ and identify its kernel and image.

# 7 Cup Products

Cup products give a pairing on cohomology groups that is natural, associative, and compatible with restriction and inflation. Note that given two $G$-modules $M$ and $N$, then $M \otimes_{\mathbb{Z}} N$ is a $G$-module via the action $g \cdot (m \otimes n) = gm \otimes gn$.

**Definition 7.1** (Cup product)**.** For $G$-modules $M$ and $N$, the *cup product* is a bilinear map

$$\smile : \mathrm{H}^p(G, M) \times \mathrm{H}^q(G, N) \longrightarrow \mathrm{H}^{p+q}(G, M \otimes_{\mathbb{Z}} N),$$

defined on cochains by

$$(f \smile g)(g_1, \ldots, g_{p+q}) := f(g_1, \ldots, g_p) \otimes g_1 \cdots g_p \cdot g(g_{p+1}, \ldots, g_{p+q}).$$

One checks this sends cocycles to cocycles and that the cup product of a cocycle with a coboundary is a coboundary, so it descends to cohomology.

*Remark* 7.2. For $p = q = 1$: given 1-cocycles $f : G \to M$ and $g : G \to N$,

$$\boxed{(f \smile g)(g_1, g_2) = f(g_1) \otimes g_1 \cdot g(g_2)}.$$

**Proposition 7.3.** *The cup product satisfies the following properties*

1. $\alpha \smile \beta = (-1)^{pq} \tau(\beta \smile \alpha)$ *where* $\tau : M \otimes N \to N \otimes M, m \otimes n \mapsto n \otimes m$.
2. $\mathrm{Res}(\alpha \smile \beta) = \mathrm{Res}(\alpha) \smile \mathrm{Res}(\beta)$.
3. $\mathrm{Inf}(\alpha \smile \beta) = \mathrm{Inf}(\alpha) \smile \mathrm{Inf}(\beta)$.
4. $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$.

**Example 7.4.** When $G$ acts trivially on $M$ and $N$, 1-cocycles are homomorphisms. For $\chi : G \to M$ and $\psi : G \to N$:

$$(\chi \smile \psi)(g_1, g_2) = \chi(g_1) \otimes \psi(g_2).$$

**Example 7.5** (Quadratic characters)**.** Let $G = G_{\mathbb{Q}}$, $M = N = \mu_2$ (trivial action), so $\mathrm{H}^1(G_{\mathbb{Q}}, \mu_2) = \mathrm{Hom}(G_{\mathbb{Q}}, \mu_2)$. Note $\mu_2 \otimes \mu_2 \cong \mathbb{Z}/2\mathbb{Z}$ via $(-1) \otimes (-1) \mapsto 1$. For squarefree $d, d'$, the cup product

$$\chi_d \smile \chi_{d'} \ \in \ \mathrm{H}^2(G_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})(= \mathrm{Br}(\mathbb{Q})[2])$$

encodes arithmetic information about the pair $(d, d')$: it is related to the Hilbert symbol $(d, d')_v$ at each place $v$ of $\mathbb{Q}$, and vanishes globally iff $(d, d')$ splits everywhere.

**Exercise 7.6.** Let $G$ be a finite group acting trivially on abelian groups $M$ and $N$.

(a) For $\chi \in \mathrm{Hom}(G, M)$ and $\psi \in \mathrm{Hom}(G, N)$, verify directly that $(\chi \smile \psi)(g_1, g_2) = \chi(g_1) \otimes \psi(g_2)$ is a 2-cocycle.

(b) Show $\chi \smile \psi$ is a coboundary if $\chi = 0$ or $\psi = 0$.

(c) For $G = \mathbb{Z}/2\mathbb{Z}$, $M = N = \mathbb{Z}/2\mathbb{Z}$: compute $\chi \smile \chi$ where $\chi : G \to \mathbb{Z}/2\mathbb{Z}$ is the unique nontrivial homomorphism. Is this zero in $\mathrm{H}^2(G, \mathbb{Z}/2\mathbb{Z})$? Use the cyclic formula to compute $\mathrm{H}^2(G, \mathbb{Z}/2\mathbb{Z})$ first.

Here we will cover some things that are useful to know for sheet 1 but not necessarily the actual lecture series.

# A Induced Modules and Shapiro's Lemma

## A.1 Group rings

**Definition A.1** (Group ring)**.** Let $G$ be a group. The *group ring* $\mathbb{Z}[G]$ is the free abelian group on the set $G$, with ring structure extending the group multiplication:

$$\mathbb{Z}[G] = \Big\{ \sum_{g \in G} n_g\, g \;\Big|\; n_g \in \mathbb{Z},\ \text{almost all } n_g = 0 \Big\}, \qquad \Big(\sum_g n_g g\Big)\Big(\sum_h m_h h\Big) = \sum_k \Big( \sum_{gh=k} n_g m_h \Big) k.$$

**Example A.2.** For $G = \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle$, one has $\mathbb{Z}[G] \cong \mathbb{Z}[X]/(X^n - 1)$ with $\sigma \mapsto X$. The *norm element* is $\mathbf{N} = 1 + \sigma + \cdots + \sigma^{n-1}$ and the *augmentation ideal* $I_G = \ker(\varepsilon : \mathbb{Z}[G] \to \mathbb{Z})$ is generated by $\sigma - 1$.

**Example A.3.** $\mathbb{Z}[G]$ is a $G$-module by left multiplication: $g \cdot (\sum n_h h) = \sum n_h (gh)$.

**Proposition A.4** (Equivalence with $\mathbb{Z}[G]$-modules)**.** *The categories of left $G$-modules and left $\mathbb{Z}[G]$-modules are isomorphic, via extending the $G$-action $\mathbb{Z}$-linearly: $(\sum n_g g) \cdot m = \sum n_g (gm)$.*

We can use *$G$-module* and *$\mathbb{Z}[G]$-module* interchangeably.

**Definition A.5** (Co-induced module)**.** For $H \leq G$ and $N$ an $H$-module, define

$$\mathrm{Ind}_H^G N := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$$

.

**Theorem A.6** (Shapiro's Lemma)**.** $\mathrm{H}^n(G, \mathrm{Ind}_H^G N) \cong \mathrm{H}^n(H, N)$ *for all $n \geq 0$.*

**Corollary A.7.** $\mathrm{H}^n(G, \mathrm{Ind}_1^G N) = 0$ *for $n \geq 1$.*

**Exercise A.8.** Let $L/K$ be a finite Galois extension, then by the normal basis theorem $L \cong K[\mathrm{Gal}(L/K)]$. Deduce that $\mathrm{H}^1(\mathrm{Gal}(L/K), L) = 0$.

**Exercise A.9.** Let $M$ be a $G$-module and $M^\dagger$ for $M$ viewed as an abelian group. We have an injection $M \hookrightarrow \mathrm{Ind}_1^G M^\dagger$. Let $Q := \mathrm{Ind}_1^G M^\dagger / M$. Show that

$$\mathrm{H}^n(G, Q) \cong \mathrm{H}^{n+1}(G, M)$$

for all $n \geq 1$.

# B The Herbrand Quotient

Let $G$ be a finite cyclic group of order $n$ and let $\sigma$ be a choice of generator. Recall that for a (finitely generated) $\mathbb{Z}[G]$-module $M$, the Herbrand quotient is

$$h^{2/1}(M) := \frac{\#\,\mathrm{H}^2(G, M)}{\#\,\mathrm{H}^1(G, M)},$$

when it exists. Note that Emerton uses the notation $h_{2/1}$. A fact used in Emerton's sheet is the following.

**Theorem B.1.** *Let $L/K$ be an extension of number fields with Galois group $G$, and $S$ be a set of places of $L$ containing the infinite ones which is preserved by $G$. Then considering $\mathcal{O}_{L,S}^\times$ as a module over $G$, we have*

$$h^{2/1}(\mathcal{O}_{L,S}^\times) = \frac{\prod_{w \in T} [L_w : K_w]}{[L : K]}.$$

*where $T$ is the set of places of $K$ under $S$, and $L_w$ denotes a choice of completion of $L$ above $w$.*

The rest of this document consists of a somewhat discursive proof. Firstly, note that Dirichlet's unit theorem furnishes us with a map of $\mathbb{Z}[G]$-modules

$$\phi : \mathcal{O}_{L,S}^\times \to \bigoplus_{v\in S} \mathbb{R}\cdot v =: V,$$

where the component at a place $v$ is

$$\phi(x)_v := \begin{cases} \log(|x|_v^2) & \text{if } v \text{ is a complex place,} \\ \log(|x|_v) & \text{else.} \end{cases}$$

Let $\Lambda := \phi(\mathcal{O}_{L,S}^\times)$ be the image, which by the unit theorem is a full rank sublattice of the sum-zero hyperplane $H := \{(z_v)_{v\in S} : \sum_{v\in S} z_v = 0\}$.

**Lemma B.2.** *If $0\to A\to B\to C\to 0$ is a short exact sequence of $\mathbb{Z}[G]$-modules, then if the Herbrand quotients of $A$ and $C$ exist then so does that of $B$ and moreover*

$$h^{2/1}(A)h^{2/1}(C) = h^{2/1}(B).$$

In particular, this will allow us to relate the Herbrand quotient of interest to that of $\Lambda$. Specifically,

$$h^{2/1}(\mathcal{O}_{L,S}^\times) = h^{2/1}(\Lambda)\, h^{2/1}(\mu(L)). \tag{1}$$

So we will study the Herbrand quotient of $\Lambda$. Moreover, let $\Lambda' := \Lambda \oplus \mathbb{Z}\cdot u$ be the span of $\Lambda$ and the vector $u = (1,\ldots,1)$ inside of $V$. Note that this is a sum of $\mathbb{Z}[G]$-modules. Then using Lemma B.2 we have $h^{2/1}(\Lambda') = h^{2/1}(\mathbb{Z})h^{2/1}(\Lambda)$, so we can update (1) to

$$h^{2/1}(\mathcal{O}_L^\times) = \frac{h^{2/1}(\mu_L)\, h^{2/1}(\Lambda')}{h^{2/1}(\mathbb{Z})}. \tag{2}$$

It would be very helpful if we could compute the Herbrand quotients of the modules which don't really seem to depend on the unit group. Then we could relate $h^{2/1}(\Lambda')$ to $h^{2/1}(\mathcal{O}_L^\times)$ explicitly.

**Lemma B.3.** *If $M$ is a finite $\mathbb{Z}[G]$-module then $h^{2/1}(M) = 1$.*

**Lemma B.4.** *If $G$ is cyclic of order $n$, $h^{2/1}(\mathbb{Z}) = n$.*

Hence, we have moved the problem to studying the lattice $\Lambda'$.

$$h^{2/1}(\mathcal{O}_{L,S}^\times) = \frac{1}{n} h^{2/1}(\Lambda'). \tag{3}$$

Pausing for a moment, why are we expecting the Herbrand quotient of $\Lambda'$ to be easier? Well that'd be the following lemma, which allows us to replace $\Lambda'$ with as simple a sublattice of $V$ as we like.

**Lemma B.5.** *Let $W$ be a finitely generated $\mathbb{R}[G]$-module, and $\Lambda_1, \Lambda_2$ be two full rank sublattices of $W$ which are $\mathbb{Z}[G]$-submodules. Then*

$$h^{2/1}(\Lambda_1) = h^{2/1}(\Lambda_2).$$

There's a natural sublattice of $V$, namely $M := \bigoplus_{v\in S}\mathbb{Z}\cdot v \subseteq V$, which looks quite approachable. Note that

$$M = \bigoplus_{w\in T}\left\{\bigoplus_{v\mid w}\mathbb{Z}\cdot v\right\} \cong \bigoplus_{w\in T} \mathrm{Ind}_{G_w}^G \mathbb{Z},$$

where $G_w$ is a choice of decomposition group at $w$ in $G$. Applying Shapiro's lemma and Lemma B.2, and then Lemma B.4 we have

$$h^{2/1}(M) = \prod_{w\in T} h^{2/1}(G_w, \mathbb{Z}) = \prod_{w\in T}[L_w : K_w],$$

where $h^{2/1}(G_w,\mathbb{Z})$ is the Herbrand quotient as $\mathbb{Z}[G_w]$-modules. Hence, to conclude, by (3) and Lemma B.5 and we have

$$h^{2/1}(\mathcal{O}_L^\times) = \frac{h^{2/1}(\Lambda')}{[L:K]} = \frac{h^{2/1}(M)}{[L:K]} = \frac{\prod_{w\in T}[L_w:K_w]}{[L:K]},$$

as claimed. $\square$

## Auxiliary Lemmata

We now prove the lemmata used above.

*Proof of Lemma B.2.* This makes use of the long-exact sequence in cohomology for cyclic groups, which forms a hexagon. See Neukirch's lectures on Class Field Theory Part I Theorem 6.4. ☐

*Proof of Lemma B.3.* We will do this directly. Firstly, let $\sigma$ be a generator of $G$ and consider the exact sequence

$$0 \to \mathrm{H}^0(G, M) \to M \xrightarrow{1-\sigma} M \to H_0(G, M) \to 0.$$

Hence, since $M$ is finite we have $\#\,\mathrm{H}^0(G, M) = \#H_0(G, M)$. Further we have an exact sequence induced by the norm $N$ given by

$$0 \to \hat{\mathrm{H}}^{-1}(G, M) \to H_0(G, M) \xrightarrow{N} \mathrm{H}^0(G, M) \to \hat{\mathrm{H}}^{0}(G, M) \to 0.$$

Hence again we have $\#\hat{\mathrm{H}}^{-1}(G, M) = \#\hat{\mathrm{H}}^{0}(G, M)$, and so since homology over cyclic groups is 2-periodic we have the claim. ☐

*Proof of Lemma B.4.* This is an immediate calculation, writing $N_G$ for the norm element for $G$:

$$\mathrm{H}^1(G, \mathbb{Z}) \cong \hat{\mathrm{H}}^{-1}(G, \mathbb{Z}) = \frac{\ker(N_G : \mathbb{Z} \to \mathbb{Z})}{(1-\sigma)\mathbb{Z}} = 0,$$

and

$$\mathrm{H}^2(G, \mathbb{Z}) \cong \hat{\mathrm{H}}^{0}(G, \mathbb{Z}) = \frac{\mathbb{Z}}{N_G(\mathbb{Z})} = \mathbb{Z}/n\mathbb{Z}.$$

☐

*Proof of Lemma B.5.* Let us first assume that $\mathbb{Q} \cdot \Lambda_1 = \mathbb{Q} \cdot \Lambda_2$. In which case, note that $\mathbb{Q} \cdot \Lambda_1/\Lambda_1 \cong \mathbb{Q}/\mathbb{Z}^r$, where $r$ is the rank of $\Lambda_1$. Since every element of this quotient is of finite order, the image of $\Lambda_2$ is a finite group and so there exists an integer $m$ such that $m\Lambda_2 \subseteq \Lambda_1$. Now consider the exact sequence

$$0 \to \Lambda_2 \xrightarrow{\times m} \Lambda_1 \to \Lambda_1/m\Lambda_2 \to 0.$$

The rightmost group is finite since $\Lambda_1$ and $\Lambda_2$ have the same rank, and so by Lemma B.2 and Lemma B.3 we have $h^{2/1}(\Lambda_1) = h^{2/1}(\Lambda_2)$.

We will now reduce the general case to the one above. Let $M_i := \mathbb{Q} \cdot \Lambda_i$. Note that

$$\mathrm{Hom}_{\mathbb{Q}[G]}(M_1, M_2) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}[G]}(M_1 \otimes_{\mathbb{Q}} \mathbb{R},\, M_2 \otimes_{\mathbb{Q}} \mathbb{R}),$$

and so since $M_1 \otimes_{\mathbb{Q}} \mathbb{R} = W = M_2 \otimes_{\mathbb{Q}} \mathbb{R}$, the right hand side has a map with determinant 1 and so so too does the left hand side. Hence there is an isomorphism $M_1 \cong M_2$. Now, under this isomorphism we can assume we're in the special case above, and so the theorem holds. ☐