Exercise sheet 4

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Exercise 1. In this exercise we explore a general recipe for proving reflection principles, which are vital for generalizing the material to 2^k-torsion of class groups for $k > 3$. The material here is inspired by Section 2.2 of `https://arxiv.org/pdf/1702.02325`.

Define $N := \mathbb{Q}_2/\mathbb{Z}_2$, which we view as a discrete $G_{\mathbb{Q}}$-module with trivial action. For every $\chi \in \mathrm{Hom}(G_{\mathbb{Q}}, \{-1, 1\})$, define $N(\chi)$ to be the "quadratic twist" of N; this is the $G_{\mathbb{Q}}$-module that has the same underlying set N but with the new action defined by

$$\sigma *_\chi n = \chi(\sigma) \cdot n.$$

Let $k \in \mathbb{Z}_{\geq 1}$ be fixed, and define the map

$$\mathrm{d}_\chi : \mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N[2^k]) \to \mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}} \times G_{\mathbb{Q}}, N[2^k])$$

given by

$$\mathrm{d}_\chi(f)(\sigma, \tau) = f(\sigma\tau) - \sigma *_\chi f(\tau) - f(\sigma).$$

Note that since N and $N(\chi)$ have the same underlying set, we have a direct identification

$$\mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N[2^k]) = \mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N(\chi)[2^k])$$

that we shall often use implicitly.

(a) Identifying $\mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N[2^k]) = \mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N(\chi)[2^k])$, check that

$$\ker(\mathrm{d}_\chi) = Z^1(G_{\mathbb{Q}}, N(\chi)[2^k]).$$

Remark. Upon unwinding the identifications, this equality is essentially by definition.

(b) Now note that as subsets of $\mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N[2^k])$, the two subspaces $Z^1(G_{\mathbb{Q}}, N(\chi_1)[2^k])$ and $Z^1(G_{\mathbb{Q}}, N(\chi_2)[2^k])$ are not equal for different χ_1, χ_2 unless $k = 1$ (convince yourself of this!). The first step of the reflection principle is to measure how far an element of $Z^1(G_{\mathbb{Q}}, N(\chi_2)[2^k])$ is off from being in $Z^1(G_{\mathbb{Q}}, N(\chi_1)[2^k])$.

Let $\psi \in Z^1(G_{\mathbb{Q}}, N(\chi_2)[2^k])$. Prove that

$$\mathrm{d}_{\chi_1}(\psi)(\sigma, \tau) = \sigma *_{\chi_1} \psi(\tau) - \sigma *_{\chi_2} \psi(\tau) = \begin{cases} 0 & \text{if } \chi_1(\sigma)\chi_2(\sigma) = 1 \\ \sigma *_{\chi_1} (2\psi(\tau)) & \text{if } \chi_1(\sigma)\chi_2(\sigma) = -1. \end{cases}$$

(c) The key aspect of the formula in (b) is that it simplifies the cocycle ψ (by doubling it). Now to take full advantage of this, we take a k-dimensional cube

$$C := \{\chi_1(1), \chi_1(2)\} \times \cdots \times \{\chi_k(1), \chi_k(2)\}.$$

Define $\rho_0 := \chi_1(1) \cdots \chi_k(1)$, and take $\rho \in C$ (viewed as a quadratic character by multiplying out the coordinates). For each $T \subseteq \{1, \ldots, k\}$, define C_T to be the subcube of $\rho \in C$ such that the i-th coordinate of ρ equals $\chi_i(2)$ for all $i \in T$. Define T_σ to be the subset of $i \in \{1, \ldots, k\}$ where $\chi_i(1)(\sigma) \cdot \chi_i(2)(\sigma) = -1$.

Prove that

$$\mathbf{1}_{\rho(\sigma)\rho_0(\sigma)=-1} = \sum_{\substack{\varnothing \neq T \subseteq T_\sigma \\ \rho \in C_T}} (-2)^{|T|-1}.$$

Hint. Define T_ρ to be the maximal subset $T \subseteq \{1, \ldots, k\}$ so that $\rho \in C_T$. Then the LHS is 1 if and only if $|T_\rho \cap T_\sigma|$ is odd. Now rewrite the RHS using this observation and the binomial theorem in the form

$$\sum_{\varnothing \neq T \subseteq S} (-2)^{|T|-1} = \sum_{i=1}^{|S|} \binom{|S|}{i} (-2)^{i-1} = \frac{1}{2}(1 - (-1)^{|S|})$$

for a suitable set S.

(d) Assume that for each $\rho \in C$, we have a cocycle $\psi_\rho \in Z^1(G_{\mathbb{Q}}, N(\rho)[2^k])$ such that

$$\sum_{\rho \in C_T} 2^{|T|} \psi_\rho = 0. \tag{0.1}$$

for every non-empty subset T of $\{1, \ldots, k\}$. Also assume that

$$\sum_{\rho \in C} 2\psi_\rho = 0. \tag{0.2}$$

Prove that

$$\sum_{\rho \in C} \psi_\rho$$

is a quadratic character, where the sum takes place in $\mathrm{Map}_{\mathrm{cnt.}}(G_{\mathbb{Q}}, N[2^k])$.

Hint. First, use equation (0.2) to deduce that $\sum_{\rho \in C} \psi_\rho$ takes values in $N[2]$. Hence observe that it suffices to prove that

$$\mathrm{d}_{\rho_0} \left(\sum_{\rho \in C} \psi_\rho \right) = 0.$$

Then rewrite this sum using part (b) as

$$\sum_{\substack{\rho \in C \\ \rho(\sigma)\rho_0(\sigma) = -1}} \sigma *_{\rho_0} (2\psi_\rho).$$

and then use part (c) to rewrite the indicator function of $\rho(\sigma)\rho_0(\sigma) = -1$, swap the order of summation and then use the assumption (0.1).

Remark. These computations can be reinterpreted as giving a way to get a non-trivial intersection between 2^∞-Hilbert class fields under suitable circumstances (namely (0.1)). This can then be used to compare various Artin pairings, essentially just by plugging in Frobenius elements in such a formal relation between cocycles.