

Exercise sheet 3

April 3, 2026

Exercise 1. In the lecture, we saw how to compute the first moment of $2\text{Cl}(K)[4]$ over imaginary quadratic fields in a filtered grid $X = X_1 \times \cdots \times X_n$. Throughout this exercise, assume (for simplicity) that every $x \in X$ is 3 modulo 4, so $\mathbb{Q}(\sqrt{-x})$ is unramified at 2.

(a) Prove that

$$\sum_{x \in X} \#(2\text{Cl}(\mathbb{Q}(\sqrt{-x}))[4])^2 = \frac{1}{4^{n-1}} \sum_{x \in X} \left(\sum_{S \subseteq [n]} \prod_{i \in S} \left(1 + \prod_{j \notin S} \left(\frac{x_j}{x_i} \right) \right) \prod_{i \notin S} \left(1 + \prod_{j \in S} \left(\frac{x_j}{x_i} \right) \right) \right)^2.$$

(b) Introduce the shorthand $x(S) := \prod_{i \in S} x_i$. Also prove that the RHS above is equal to

$$\frac{1}{4^{n-1}} \sum_{\substack{S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 = [n] \\ T_1 \sqcup T_2 \sqcup T_3 \sqcup T_4 = [n]}} \sum_{x \in X} \left(\frac{x(S_3)x(S_4)}{x(S_1)} \right) \left(\frac{x(S_1)x(S_2)}{x(S_4)} \right) \left(\frac{x(T_3)x(T_4)}{x(T_1)} \right) \left(\frac{x(T_1)x(T_2)}{x(T_4)} \right),$$

where \sqcup denotes disjoint union. Then prove that the RHS above is also equal to

$$\frac{1}{4^{n-1}} \sum_{(S_{\mathbf{v}})_{\mathbf{v} \in (\mathbb{F}_2^2)^2}} \sum_{\text{partition of } [n]} \sum_{x \in X} \prod_{\substack{\mathbf{v} \in (\mathbb{F}_2^2)^2 \\ \mathbf{w} \in (\mathbb{F}_2^2)^2}} \left(\frac{x(S_{\mathbf{v}})}{x(S_{\mathbf{w}})} \right)^{\psi(\mathbf{v}, \mathbf{w})},$$

where

$$\psi(\mathbf{v}, \mathbf{w}) = (v_1 + w_1)(v_1 + w_2) + (v_3 + w_3)(v_3 + w_4).$$

Hint. For the second part of the exercise, biject \mathbb{F}_2^2 with $\{1, 2, 3, 4\}$ by sending $(0, 0)$ to 1, $(0, 1)$ to 2, $(1, 0)$ to 3 and $(1, 1)$ to 4 (call this bijection B), and define $S_{B^{-1}(i), B^{-1}(j)}$ to be the intersection of $S_i \cap T_j$.

(c) Describe the linked indices in this case, and the maximal unlinked sets.

Bonus. The higher moments can be treated similarly. For details, take a look at §5.5 and §5.6 in the Fouvry–Klüners Inventiones paper [here](#).

In general, it is non-trivial to go back from moments to the distribution. The argument of Fouvry–Klüners gives no error term (this argument is in a separate companion paper), but there is a neat argument in Proposition 10.5 of <https://arxiv.org/pdf/2207.05143>.

Exercise 2. The following technique is known as “repeated Cauchy–Schwarz”. It is used often in the circle method (Weyl differencing) and additive combinatorics. Its power lies in the fact that it allows one to create more variables. In our setting, a small variant of it will be a key tool as well: it will allow us to create the comparisons between class groups in a very neat way.

Define $[n] := \{1, \dots, n\}$. Let X_1, \dots, X_n be finite sets, and let $F : X_1 \times \dots \times X_n \rightarrow \mathbb{R}$ be a function. Then we have

$$\left| \sum_{(a_1, \dots, a_n) \in X_1 \times \dots \times X_n} F(a_1, \dots, a_n) \right|^{2^n} \leq \prod_{i=1}^n |X_i|^{2^n - 2} \cdot \sum_{\substack{(a_{1,1}, \dots, a_{n,1}) \in X_1 \times \dots \times X_n \\ (a_{1,2}, \dots, a_{n,2}) \in X_1 \times \dots \times X_n}} \prod_{g \in \text{Map}([n], [2])} F(a_{1,g(1)}, \dots, a_{n,g(n)}).$$

Hint. First prove the following inequality by using a single application of Cauchy–Schwarz:

$$\left(\sum_{x \in X} \left| \sum_{y \in Y} F(x, y) \right| \right)^2 \leq |X| \cdot \sum_{x \in X} \sum_{\substack{y_1 \in Y \\ y_2 \in Y}} F(x, y_1) F(x, y_2).$$

Then use the above inequality repeatedly to solve the exercise.

Important bonus question. Define for each $j \in [n]$ the product space

$$Y_j := X_1 \times \dots \times X_{j-1} \times X_{j+1} \times \dots \times X_n,$$

so X_j is omitted. Let $g : Y_j \rightarrow [-1, 1]$ be a function, which we view as a function on $X_1 \times \dots \times X_n$ via projection. What can you say about

$$\left| \sum_{(a_1, \dots, a_n) \in X_1 \times \dots \times X_n} F(a_1, \dots, a_n) \prod_{j=1}^n g_j(a_1, \dots, a_n) \right|^{2^n} ?$$

Now let $g : X_1 \times \dots \times X_n \rightarrow [-1, 1]$ be arbitrary (so does not necessarily factor through one of the coordinates). What can you say about

$$\left| \sum_{(a_1, \dots, a_n) \in X_1 \times \dots \times X_n} F(a_1, \dots, a_n) g(a_1, \dots, a_n) \right|^{2^n} ?$$