

Exercise sheet 2

April 5, 2026

Exercise 1. Let K/\mathbb{Q} be Galois with Hilbert class field H_K .

- (a) Prove that H_K is Galois over \mathbb{Q} .
- (b) From (a) and Galois theory, we have an exact sequence

$$1 \rightarrow \text{Gal}(H_K/K) \rightarrow \text{Gal}(H_K/\mathbb{Q}) \rightarrow \text{Gal}(K/\mathbb{Q}) \rightarrow 1.$$

Then $\text{Gal}(K/\mathbb{Q})$ acts on $\text{Gal}(H_K/K)$ by “lift-and-conjugate” (more generally, this works for any short exact sequence with abelian kernel): given an automorphism σ of K , you can lift it to an automorphism $\tilde{\sigma}$ of H_K , and then you can let $\tilde{\sigma}$ act on elements of $\text{Gal}(H_K/K)$ by conjugating with $\tilde{\sigma}$. Prove that “lift-and-conjugate” is indeed a well-defined action.

- (c) Let $K \subseteq F \subseteq H_K$. Recall that $\text{Gal}(H_K/F)$ is a subgroup of $\text{Gal}(H_K/K)$ by Galois theory. Prove that F is Galois over \mathbb{Q} if and only if $\text{Gal}(H_K/F)$ is stable (not necessarily fixed!) under the action from part (b).
- (d) Prove that there is a natural action of $\text{Gal}(K/\mathbb{Q})$ on $\text{Cl}(K)$.
- (e) Prove that there is a $\text{Gal}(K/\mathbb{Q})$ -equivariant isomorphism $\text{Cl}(K) \rightarrow \text{Gal}(H_K/K)$.

Hint. In order to prove the existence of the above isomorphism, you will need to use one of the main theorems of class field theory.

Remark. One of the most important points of class field theory is not just the existence of key isomorphisms, but that these are given explicitly with good functorial properties.

- (f) Now assume moreover that K is quadratic. Prove that the exact sequence from (b) admits a splitting $\text{Gal}(K/\mathbb{Q}) \rightarrow \text{Gal}(H_K/\mathbb{Q})$.

Hint. Use that there is a ramified prime in K , and look at its inertia subgroup inside $\text{Gal}(H_K/\mathbb{Q})$ to produce a splitting.

- (g) Still assume that K is quadratic. Prove that $\text{Gal}(K/\mathbb{Q}) \cong C_2$ acts on $\text{Cl}(K)$ by inversion (i.e. the unique non-trivial element of C_2 sends a class x to its inverse $-x$). Combine this with the previous parts to deduce that $\text{Gal}(K/\mathbb{Q})$ also acts on $\text{Gal}(H_K/K)$ by inversion, and hence that

$$\text{Gal}(H_K/\mathbb{Q}) \cong \text{Cl}(K) \rtimes C_2$$

with C_2 acting by inversion.

Exercise 2. Recall that $A^\vee := \text{Hom}(A, \mathbb{C}^*)$.

(a) By class field theory we have canonical isomorphisms

$$\text{Cl}(K)^\vee \cong \text{Gal}(H_K/K)^\vee \cong \text{Hom}_{\text{ur}}(G_K, \mathbb{C}^*). \quad (0.1)$$

Either give or look up what these canonical isomorphisms are.

(b) Prove that under the identifications in equation (0.1), the canonical duality pairing is explicitly given by

$$\begin{array}{ccccc} \text{Cl}(K) & \times & \text{Cl}(K)^\vee & \rightarrow & \mathbb{C}^* \\ \downarrow = & & \downarrow \cong & & \downarrow = \\ \text{Cl}(K) & \times & \text{Hom}_{\text{ur}}(G_K, \mathbb{C}^*) & \rightarrow & \mathbb{C}^* \end{array}$$

where the bottom pairing is $(\mathfrak{p}, \chi) \mapsto \chi(\text{Frob } \mathfrak{p})$.

(c) Can you make a similar diagram for the k -th Artin pairing

$$\text{Art}_k : 2^{k-1}\text{Cl}(K)[2^k] \times 2^{k-1}\text{Cl}(K)^\vee[2^k] \rightarrow \{\pm 1\}?$$

Exercise 3. Compute the Rédei matrices of $\mathbb{Q}(\sqrt{-15})$ and $\mathbb{Q}(\sqrt{-355})$. Then use genus theory from Exercise set 1 to define Rédei matrices for even discriminants, and compute the Rédei matrices of $\mathbb{Q}(\sqrt{-26})$ and $\mathbb{Q}(\sqrt{-105})$.

What is $\dim_{\mathbb{F}_2} 2(\text{Cl}(K)[4])$ for $K \in \{\mathbb{Q}(\sqrt{-15}), \mathbb{Q}(\sqrt{-26}), \mathbb{Q}(\sqrt{-105}), \mathbb{Q}(\sqrt{-355})\}$? Check your answer with the LMFDB.